Discrete Mathematics

Discrete mathematics is the branch of mathematics that involves finite or discontinuous quantities. In this unit, you will learn about sequences, series, probability, and statistics.

UNIT



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Chapter 11 Sequences and Series

Chapter 12 *Probability and <u>Statistics</u>*

CONTENTS

Web uest Internet Project

'Minesweeper': Secret to Age-Old Puzzle?

Source: USA TODAY, November 3, 2000

"Minesweeper, a seemingly simple game included on most personal computers, could help mathematicians crack one of the field's most intriguing problems. The buzz began after Richard Kaye, a mathematics professor at the University of Birmingham in England, started playing *Minesweeper*. After playing the game steadily for a few weeks, Kaye realized that *Minesweeper*, if played on a much larger grid, has the same mathematical characteristics as other problems deemed insolvable." In this project, you will research a mathematician of the past and his or her role in the development of discrete mathematics.



Log on to **www.algebra2.com/webquest**. Begin your WebQuest by reading the Task.

Then continue working on your WebQuest as you study Unit 4.

🔆 Mega Minesweeper

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CONTENTS



Chapter **D** Sequences and Series

What You'll Learn

- **Lessons 11-1 through 11-5** Use arithmetic and geometric sequences and series.
- **Lesson 11-6** Use special sequences and iterate functions.
- **Lesson 11-7** Expand powers by using the Binomial Theorem.
- **Lesson 11-8** Prove statements by using mathematical induction.

Why It's Important

Many number patterns found in nature and used in business can be modeled by sequences, which are lists of numbers. Some sequences are classified by the method used to predict the next term from the previous term(s). When the terms of a sequence are added, a series is formed. *In Lesson 11-2, you will learn how the number of seats in the rows of an amphitheater can be modeled using a series.*

CONTENTS

Key Vocabulary

- arithmetic sequence (p. 578)
- arithmetic series (p. 583)
- sigma notation (p. 585)
- geometric sequence (p. 588)
- geometric series (p. 594)

Getting Started

Prerequisite Skills To be successful in this chapter, you'll need to master these skills and be able to apply them in problem-solving situations. Review these skills before beginning Chapter 11.

For Lessons 11-1 and 11-3

Solve each equation. (For review, see Lessons 1-3 and 5-5.)

1.	36 = 12 + 4x	2.	-40 = 10 + 5x
3.	12 - 3x = 27	4.	$162 = 2x^4$
5.	$\frac{1}{8} = 4x^5$	6.	$3x^3 + 4 = -20$

For Lessons 11-1 and 11-5

Graph each function. (For review, see Lesson 2-1.)

- **7.** $\{(1, 1), (2, 3), (3, 5), (4, 7), (5, 9)\}$
- **7.** {(1, 1), (2, 3), (3, 5), (4, 7), (5, 9)}
 8. {(1, -20), (2, -16), (3, -12), (4, -8), **8.** {(1, -20), (2, -16), (3, -12), (4, -8), (5, -4)}

For Lessons 11-1 through 11-5, 11-8

Evaluate each expression for the given value(s) of the variable(s). (For review, see Lesson 1-1.) **12.** $\frac{x}{2}(y+z)$ if x = 10, y = 3, and z = 25**11.** x + (y - 1)z if x = 3, y = 8, and z = 2**13.** $a \cdot b^{c-1}$ if $a = 2, b = \frac{1}{2}$, and c = 7**14.** $\frac{a(1-bc)^2}{1-b}$ if a = -2, b = 3, and c = 5**16.** $\frac{n(n+1)}{2}$ if n = 10**15.** $\frac{a}{1-b}$ if $a = \frac{1}{2}$, and $b = \frac{1}{6}$



Make this Foldable to record information about sequences and series. Begin with one sheet of 11" by 17" paper and four sheets of notebook paper.





Staple and Label

Fold the notebook paper in half lengthwise. Insert two sheets of notebook paper in each tab and staple edges. Label with lesson numbers.

Reading and Writing As you read and study the chapter, fill the journal with examples for each lesson.

CONTENTS



Evaluate Expressions

Solve Equations

11-1

Arithmetic Sequences

What You'll Learn

- Use arithmetic sequences.
- Find arithmetic means.

Vocabulary

- sequence
- term
- arithmetic sequence
- common difference
- arithmetic means

Study Tip

Sequences

The numbers in a sequence may not be ordered. For example, the numbers 33, 25, 36, 40, 36, 66, 63, 50, ... are a sequence that represents the number of home runs Sammy Sosa hit in each year beginning with 1993.

How are arithmetic sequences related to roofing?

A roofer is nailing shingles to the roof of a house in overlapping rows. There are three shingles in the top row. Since the roof widens from top to bottom, one additional shingle is needed in each successive row.

Chingles 2 4 5 6 7	Row	1	2	3	4	5	6	7
Shingles 3 4 5 6 7	Shingles	3	4	5	6	7	8	9

ARITHMETIC SEQUENCES The numbers 3, 4, 5, 6, ..., representing the number of shingles in each row, are an example of a sequence of numbers. A **sequence** is a list of numbers in a particular order. Each number in a sequence is called a **term**. The first term is symbolized by a_1 , the second term is symbolized by a_2 , and so on.

The graph represents the information from the table above. A sequence is a function whose domain is the set of positive integers. You can see from the graph that a sequence is a discrete function.

Many sequences have patterns. For example, in the sequence above for the number of shingles, each term can be found by adding 1 to the previous term. A



sequence of this type is called an arithmetic sequence. An **arithmetic sequence** is a sequence in which each term after the first is found by adding a constant, called the **common difference** *d*, to the previous term.

Example 1) Find the Next Terms

Find the next four terms of the arithmetic sequence 55, 49, 43,

Find the common difference *d* by subtracting two consecutive terms.

49 - 55 = -6 and 43 - 49 = -6 So, d = -6.

Now add -6 to the third term of the sequence, and then continue adding -6 until the next four terms are found.

$$43 \underbrace{0}_{+(-6)} 37 \underbrace{0}_{+(-6)} 31 \underbrace{0}_{+(-6)} 25 \underbrace{0}_{+(-6)} 19$$

The next four terms of the sequence are 37, 31, 25, and 19.

There is a pattern in the way the terms of an arithmetic sequence are formed. It is possible to develop a formula for each term of an arithmetic sequence in terms of the first term a_1 and the common difference *d*. Look at the sequence in Example 1.



Saguanaa	numbers	55	49	43	37	
Sequence	symbols	a ₁	a ₂	a ₃	a ₄	 a _n
Expressed in	numbers	55 + 0(-6)	55 + 1(-6)	55 + 2(-6)	55 + 3(-6)	 55 + (n - 1)(-6)
the First Term	symbols	$a_1 + 0 \cdot d$	$a_1 + 1 \cdot d$	$a_1 + 2 \cdot d$	$a_1 + 3 \cdot d$	 $a_1 + (n-1)d$

The following formula generalizes this pattern for any arithmetic sequence.

Key Concept

nth Term of an Arithmetic Sequence

The *n*th term a_n of an arithmetic sequence with first term a_1 and common difference *d* is given by

$$a_n = a_1 + (n - 1)d$$
,

where *n* is any positive integer.

More About. . .



Construction •······

The table below shows typical costs for a construction company to rent a crane for one, two, three, or four months.

Months	Cost (\$)
1	75,000
2	90,000
3	105,000
4	120,000

Source: www.howstuffworks.com

Study Tip

Arithmetic

Sequences

always linear.

An equation for an arithmetic sequence is

Example 2) Find a Particular Term

•**CONSTRUCTION** Refer to the information at the left. Assuming that the arithmetic sequence continues, how much would it cost to rent the crane for twelve months?

- **Explore** Since the difference between any two successive costs is \$15,000, the costs form an arithmetic sequence with common difference 15,000.
- **Plan** You can use the formula for the *n*th term of an arithmetic sequence with $a_1 = 75,000$ and d = 15,000 to find a_{12} , the cost for twelve months.
- Solve $a_n = a_1 + (n-1)d$ Formula for *n*th term $a_{12} = 75,000 + (12-1)15,000$ $n = 12, a_1 = 75,000, d = 15,000$ $a_{12} = 240,000$ Simplify.It would cost \$240,000 to rent the crane for twelve months.
- **Examine** You can find terms of the sequence by adding 15,000. a_5 through a_{12} are 135,000, 150,000, 165,000, 180,000, 195,000, 210,000, 225,000, and 240,000. Therefore, \$240,000 is correct.

Example 3 Write an Equation for the nth Term

Write an equation for the *n*th term of the arithmetic sequence 8, 17, 26, 35, In this sequence, $a_1 = 8$ and d = 9. Use the *n*th term formula to write an equation.

 $a_n = a_1 + (n - 1)d$ Formula for *n*th term $a_n = 8 + (n - 1)9$ $a_1 = 8, d = 9$ $a_n = 8 + 9n - 9$ Distributive Property $a_n = 9n - 1$ Simplify.

An equation is $a_n = 9n - 1$.

www.algebra2.com/extra_examples



Lesson 11-1 Arithmetic Sequences 579

Algebra Activity

Arithmetic Sequences

Study the figures below. The length of an edge of each cube is 1 centimeter.



Model and Analyze

- 1. Based on the pattern, draw the fourth figure on a piece of isometric dot paper.
- **2.** Find the volumes of the four figures.
- **3.** Suppose the number of cubes in the pattern continues. Write an equation that gives the volume of Figure *n*.
- 4. What would the volume of the twelfth figure be?

ARITHMETIC MEANS Sometimes you are given two terms of a sequence, but they are not successive terms of that sequence. The terms between any two nonsuccessive terms of an arithmetic sequence are called **arithmetic means**. In the sequence below, 41, 52, and 63 are the three arithmetic means between 30 and 74.

19, 30, 41, 52, 63, 74, 85, 96, ...

3 arithmetic means between 30 and 74

Example 4) Find Arithmetic Means

Find the four arithmetic means between 16 and 91.

You can use the *n*th term formula to find the common difference. In the sequence 16, ?, ?, ?, ?, 91, ..., a_1 is 16 and a_6 is 91.

 $a_n = a_1 + (n - 1)d$ Formula for the *n*th term $a_6 = 16 + (6 - 1)d$ $n = 6, a_1 = 16$ 91 = 16 + 5d*a*₆ = 91 75 = 5dSubtract 16 from each side. 15 = dDivide each side by 5.

Now use the value of *d* to find the four arithmetic means.

$$16 \underbrace{31}_{+15} \underbrace{46}_{+15} \underbrace{61}_{+15} \underbrace{76}_{+15}$$

The arithmetic means are 31, 46, 61, and 76. **CHECK** $76 + 15 = 91 \sqrt{100}$

Check for Understanding

Concept Check	1. Explain why the sequence 4, 5, 7, 10, 14, is not arithmetic.						
·	2. Find the 15th term in the arithmetic set	equence -3, 4, 11, 18,					
	3. OPEN ENDED Write an arithmetic sequence with common difference -5 .						
Guided Practice	Find the next four terms of each arithm	etic sequence.					
	4. 12, 16, 20,	5. 3, 1, -1,					
	Find the first five terms of each arithmetic sequence described.						
	6. $a_1 = 5, d = 3$ 7. $a_1 = 14, d = -2$						

or *d* = 15.

is d = 91 - (16 + 4d)

You may prefer this method. The four means

Study Tip

Alternate Method

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8. Find a_{13} for the arithmetic sequence $-17, -12, -7, \dots$

Find the indicated term of each arithmetic sequence.

- **10.** $a_1 = -5, d = 7, n = 13$ **9.** $a_1 = 3, d = -5, n = 24$
- **11.** Complete: 68 is the ? th term of the arithmetic sequence $-2, 3, 8, \ldots$
- **12.** Write an equation for the *n*th term of the arithmetic sequence -26, -15, -4, 7, ...
- 13. Find the three arithmetic means between 44 and 92.
- Application 14. ENTERTAINMENT A basketball team has a halftime promotion where a fan gets to shoot a 3-pointer to try to win a jackpot. The jackpot starts at \$5000 for the first game and increases \$500 each time there is no winner. Ken has tickets to the fifteenth game of the season. How much will the jackpot be for that game if no one wins by then?

Practice and Apply

Homework Help

For Exercises	See Examples
15-28, 49	1
29-45, 51	2
46-48, 50	3
52-55	4

Extra Practice See page 851.



Tower of Pisa •

Upon its completion in 1370, the Leaning Tower of Pisa leaned about 1.7 meters from vertical. Today, it leans about 5.2 meters from vertical. Source: Associated Press

Find the next four terms of each arithmetic sequence.

16. 31, 24, 17,
18. -8, -5, -2,
20. $\frac{18}{5}, \frac{16}{5}, \frac{14}{5}, \dots$
22. 1.3, 3.8, 6.3,

Find the first five terms of each arithmetic sequence described.

23. $a_1 = 2, d = 13$	24. $a_1 = 41, d = 5$
25. $a_1 = 6, d = -4$	26. $a_1 = 12, d = -3$
27. $a_1 = \frac{4}{3}, d = -\frac{1}{3}$	28. $a_1 = \frac{5}{8}, d = \frac{3}{8}$

29. Find a_8 if $a_n = 4 + 3n$.

30. If $a_n = 1 - 5n$, what is a_{10} ?

Find the indicated term of each arithmetic sequence.

31. $a_1 = 3, d = 7, n = 14$	32. $a_1 = -4, d = -9, n = 20$
33. $a_1 = 35, d = 3, n = 101$	34. $a_1 = 20, d = 4, n = 81$
35. $a_1 = 5, d = \frac{1}{3}, n = 12$	36. $a_1 = \frac{5}{2}, d = -\frac{3}{2}, n = 11$
37. <i>a</i> ₁₂ for -17, -13, -9,	38. <i>a</i> ₁₂ for 8, 3, −2, …
39. <i>a</i> ₂₁ for 121, 118, 115,	40. <i>a</i> ₄₃ for 5, 9, 13, 17, …

- **41. GEOLOGY** Geologists estimate that the continents of Europe and North America are drifting apart at a rate of an average of 12 miles every 1 million years, or about 0.75 inch per year. If the continents continue to drift apart at that rate, how many inches will they drift in 50 years? (*Hint*: $a_1 = 0.75$)
- •• 42. TOWER OF PISA To prove that objects of different weights fall at the same rate, Galileo dropped two objects with different weights from the Leaning Tower of Pisa in Italy. The objects hit the ground at the same time. When an object is dropped from a tall building, it falls about 16 feet in the first second, 48 feet in the second second, and 80 feet in the third second, regardless of its weight. How many feet would an object fall in the tenth second?



Complete the statement for each arithmetic sequence.

- **43.** 170 is the <u>?</u> term of -4, 2, 8,
- **44.** 124 is the _____ term of −2, 5, 12, … .
- **45.** -14 is the _? term of $2\frac{1}{5}$, 2, $1\frac{4}{5}$, ...

Write an equation for the *n*th term of each arithmetic sequence.

46. 7, 16, 25, 34, ... **47.** 18, 11, 4, -3, ... **48.** -3, -5, -7, -9, ...

GEOMETRY For Exercises 49–51, refer to the first three arrays of numbers below.

- **49.** Make drawings to find the next three numbers in this pattern.
- **50.** Write an equation representing the *n*th number in this pattern.
- **51.** Is 397 a number in this pattern? Explain.



D 11

67. 4m + 7; m = 1, 2, 3, 4, 5

Find the arithmetic means in each sequence.

52.	55, _	?_,_	?_,_	?_,	115		53.	10,	<u>?</u> ,	_?_,	-8			
54.	-8,	?_,	?,	?	, ?	_,7	55.	3, _	?_,_	?,	?,	?,	?	_, 27

- **56. CRITICAL THINKING** The numbers *x*, *y*, and *z* are the first three terms of an arithmetic sequence. Express *z* in terms of *x* and *y*.
- **57.** WRITING IN MATH Answer the question that was posed at the beginning of the lesson.

How are arithmetic sequences related to roofing?

Include the following in your answer:

- the words that indicate that the numbers of shingles in the rows form an arithmetic sequence, and
- explanations of at least two ways to find the number of shingles in the fifteenth row.



58. What number follows 20 in this arithmetic sequence? 8, 11, 14, 17, 20, ... (A) 5 (B) 23 (C) 26 (D) 29 59. Find the first term in the arithmetic sequence. $\dots, 8\frac{1}{3}, 7, 5\frac{2}{3}, 4\frac{1}{3}, \dots$

B $9\frac{2}{3}$ **C** $10\frac{1}{3}$

65. 3n - 1; n = 1, 2, 3, 4

Maintain Your Skills

Mixed Review 60. COMPUTERS Suppose a computer that costs \$3000 new is only worth \$600 after 3 years. What is the average annual rate of depreciation? (Lesson 10-6)

Solve each equation.(Lesson 10-5)61. $3e^x - 2 = 0$ 62. $e^{3x} = 4$ 63. $\ln(x + 2) = 5$

64. If *y* varies directly as *x* and y = 5 when x = 2, find *y* when x = 6. (Lesson 9-4)

66. 6 - i; i = 1, 2, 3, 4

Getting Ready for PREREQUISITE SKILL Evaluate each expression for the given values of the the Next Lesson variable. (To review evaluating expressions, see Lesson 1-1.)

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11-2 Arithmetic Series

What You'll Learn

Find sums of arithmetic series.

A

Use sigma notation.

Vocabulary

- series
- arithmetic series
- sigma notation
- index of summation

•

do arithmetic series apply to amphitheaters? How

The first amphitheaters were built for contests between gladiators. Modern amphitheaters are usually used for the performing arts. Amphitheaters generally get wider as the distance from the stage increases. Suppose a small amphitheater can seat 18 people in the first row and each row can seat 4 more people than the previous row.



Study Tip

Indicated Sum

The sum of a series is the result when the terms of the series are added. An indicated sum is the expression that illustrates the series, which includes the terms + or -.

ARITHMETIC SERIES The numbers of seats in the rows of the amphitheater form an arithmetic sequence. To find the number of people who could sit in the first four rows, add the first four terms of the sequence. That sum is 18 + 22 + 26 + 30 or 96. A series is an indicated sum of the terms of a sequence. Since 18, 22, 26, 30 is an arithmetic sequence, 18 + 22 + 26 + 30 is an **arithmetic series**. Below are some more arithmetic sequences and the corresponding arithmetic series.

rithmetic Sequence	Arithmetic Series
5, 8, 11, 14, 17	5 + 8 + 11 + 14 + 17
-9, -3, 3	-9 + (-3) + 3
$\frac{3}{8}, \frac{8}{8}, \frac{13}{8}, \frac{18}{8}$	$\frac{3}{8} + \frac{8}{8} + \frac{13}{8} + \frac{18}{8}$

 S_n represents the sum of the first *n* terms of a series. For example, S_4 is the sum of the first four terms. For the series 5 + 8 + 11 + 14 + 17, S_4 is 5 + 8 + 11 + 14 or 38.

To develop a formula for the sum of any arithmetic series, consider the series below.

 $S_{\rm q} = 4 + 11 + 18 + 25 + 32 + 39 + 46 + 53 + 60$

Suppose we write S_0 in two different orders and add the two equations.

 $S_{q} = 4 + 11 + 18 + 25 + 32 + 39 + 46 + 53 + 60$ (+) $S_9 = 60 + 53 + 46 + 39 + 32 + 25 + 18 + 11 + 4$ $2S_9 = 9(64)$ $S_9 = \frac{9}{2}(64)$ Note that the sum had 9 terms. The first and last terms of the sum are 64.

An arithmetic sequence S_n has n terms, and the sum of the first and last terms is $a_1 + a_n$. Thus, the formula $S_n = \frac{n}{2}(a_1 + a_n)$ represents the sum of any arithmetic series.



Example **1)** Find the Sum of an Arithmetic Series

Find the sum of the first 100 positive integers.

The series is 1 + 2 + 3 + ... + 100. Since you can see that $a_1 = 1$, $a_{100} = 100$, and d = 1, you can use either sum formula for this series.

Method 1		Method 2
$S_n = \frac{n}{2}(a_1 + a_n)$	Sum formula	$S_n = \frac{n}{2} [2a_1 + (n-1)d]$
$S_{100} = \frac{100}{2}(1+100)$	$n = 100, a_1 = 1, a_{100} = 100, d = 1$	$S_{100} = \frac{100}{2} [2(1) + (100 - 1)1]$
$S_{100} = 50(101)$	Simplify.	$S_{100} = 50(101)$
$S_{100} = 5050$	Multiply.	$S_{100} = 5050$
The sum of the first 10	0 positivo integora is 5050	

The sum of the first 100 positive integers is 5050.

Example 2 Find the First Term

• **RADIO** A radio station considered giving away \$4000 every day in the month of August for a total of \$124,000. Instead, they decided to increase the amount given away every day while still giving away the same total amount. If they want to increase the amount by \$100 each day, how much should they give away the first day?

You know the values of n, S_n , and d. Use the sum formula that contains d.

$S_n = \frac{n}{2} [2a_1 + (n-1)d]$	Sum formula
$S_{31} = \frac{31}{2} [2a_1 + (31 - 1)100]$	<i>n</i> = 31, <i>d</i> = 100
$124,000 = \frac{31}{2}(2a_1 + 3000)$	$S_{31} = 124,000$
$8000 = 2a_1 + 3000$	Multiply each side by $\frac{2}{31}$.
$5000 = 2a_1$	Subtract 3000 from each side.
$2500 = a_1$	Divide each side by 2.
TTI 1 1 11.	

The radio station should give away \$2500 the first day.

Sometimes it is necessary to use both a sum formula and the formula for the *n*th term to solve a problem.

Example **3** Find the First Three Terms

Find the first three terms of an arithmetic series in which $a_1 = 9$, $a_n = 105$, and $S_n = 741$. **Step 1** Since you know a_1, a_n , and S_n , Step 2 Find *d*. use $S_n = \frac{n}{2}(a_1 + a_n)$ to find *n*. $a_n = a_1 + (n-1)d$ $S_n = \frac{n}{2}(a_1 + a_n)$ 105 = 9 + (13 - 1)d96 = 12d $741 = \frac{n}{2}(9 + 105)$ 8 = d741 = 57n13 = n**Step 3** Use *d* to determine a_2 and a_3 . $a_2 = 9 + 8 \text{ or } 17$ $a_3 = 17 + 8 \text{ or } 25$ The first three terms are 9, 17, and 25.



Radio •·····

99.0% of teens ages 12–17 listen to the radio at least once a week. 79.1% listen at least once a day. **Source:** Radio Advertising Bureau



Study Tip

Sigma Notation There are many ways to

represent a given series.

$$\sum_{r=4}^{5} (r-3)$$

= $\sum_{s=2}^{7} (s-1)$
= $\sum_{j=0}^{5} (j+1)$

SIGMA NOTATION Writing out a series can be time-consuming and lengthy. For convenience, there is a more concise notation called **sigma notation**. The series

3 + 6 + 9 + 12 + ... + 30 can be expressed as $\sum_{n=1}^{10} 3n$. This expression is read *the sum* of 3n as n goes from 1 to 10.



The variable, in this case *n*, is called the **index of summation**.

To generate the terms of a series given in sigma notation, successively replace the index of summation with consecutive integers between the first and last values of the index, inclusive. For the series above, the values of *n* are 1, 2, 3, and so on, through 10.



$$\sum_{j=5}^{8} (3j-4) = [3(5) - 4] + [3(6) - 4] + [3(7) - 4] + [3(8) - 4] = 11 + 14 + 17 + 20 = 62$$

Method 2

Since the sum is an arithmetic series, use the formula $S_n = \frac{n}{2}(a_1 + a_n)$. There are 4 terms, $a_1 = 3(5) - 4$ or 11, and $a_4 = 3(8) - 4$ or 20. $S_4 = \frac{4}{2}(11 + 20)$ $S_4 = 62$

The sum of the series is 62.

You can use the sum and sequence features on a graphing calculator to find the sum of a series.

Study Tip

Graphing Calculators

On the TI-83 Plus, sum(is located on the LIST MATH menu. The function seq(is located on the LIST OPS menu.



Sums of Series

The calculator screen shows the evaluation of

 $\sum_{N=2}^{10} (5N - 2)$. The first four entries for seq(are

- the formula for the general term of the series,
- the index of summation,
- the first value of the index, and
- the last value of the index, respectively.

The last entry is always 1 for the types of series that we are considering.

Think and Discuss

- **1.** Explain why you can use any letter for the index of summation.
- **2.** Evaluate $\sum_{n=1}^{8} (2n-1)$ and $\sum_{j=5}^{12} (2j-9)$. **Make a conjecture** as to their

relationship and explain why you think it is true.

www.algebra2.com/extra_examples



sum(seq(5N-2,N,2 ,10,1))

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Check for Understanding

Concept Check 1. Explain the difference between a sequence and a series.

- **2. OPEN ENDED** Write an arithmetic series for which $S_5 = 10$.
- **3. OPEN ENDED** Write the series 7 + 10 + 13 + 16 using sigma notation.

Guided Practice Find S_n for each arithmetic series described.

4. $a_1 = 4, a_n = 100, n = 25$ **5.** $a_1 = 40, n = 20, d = -3$ **6.** $a_1 = 132, d = -4, a_n = 52$ **7.** $d = 5, n = 16, a_n = 72$

Find the sum of each arithmetic series.

8. 5 + 11 + 17 + ... + 959. 38 + 35 + 32 + ... + 210. $\sum_{n=1}^{7} (2n+1)$ 11. $\sum_{k=3}^{7} (3k+4)$

Find the first three terms of each arithmetic series described.

- **12.** $a_1 = 11, a_n = 110, S_n = 726$ **13.** $n = 8, a_n = 36, S_n = 120$
- Application14. WORLD CULTURES The African-American festival of *Kwanzaa* includes a ritual involving candles. The first night, a candle is lit and then blown out. The second night, a new candle and the candle from the previous night are lit and blown out. This pattern of lighting a new candle and relighting all the candles from the previous nights is continued for seven nights. Use a formula from this lesson to find the total number of candle lightings during the festival.

Practice and Apply

Homework Help

For Exercises	See Examples	15.
15–32, 39,	1, 2	17.
40.45		

4

3

Extra Practice

See page 851.

33-38

41-44

Find S_n for each arithmetic series described.

15. $a_1 = 7, a_n = 79, n = 8$	16. $a_1 = 58, a_n = -7, n = 26$
17. $a_1 = 43, n = 19, a_n = 115$	18. $a_1 = 76, n = 21, a_n = 176$
19. $a_1 = 7, d = -2, n = 9$	20. $a_1 = 3, d = -4, n = 8$
21. $a_1 = 5, d = \frac{1}{2}, n = 13$	22. $a_1 = 12, d = \frac{1}{3}, n = 13$
23. $d = -3, n = 21, a_n = -64$	24. $d = 7, n = 18, a_n = 72$
25. $d = \frac{1}{5}, n = 10, a_n = \frac{23}{10}$	26. $d = -\frac{1}{4}$, $n = 20$, $a_n = -\frac{53}{12}$

27. TOYS Jamila is making a triangular wall with building blocks. The top row has one block, the second row has three, the third has five, and so on. How many rows can she make with a set of 100 blocks?



28. CONSTRUCTION A construction company will be fined for each day it is late completing its current project. The daily fine will be \$4000 for the first day and will increase by \$1000 each day. Based on its budget, the company can only afford \$60,000 in total fines. What is the maximum number of days it can be late?

Find the sum of each arithmetic series.



- 39. Find the sum of the first 1000 positive even integers.
- **40.** What is the sum of the multiples of 3 between 3 and 999, inclusive?

Find the first three terms of each arithmetic series described.

41. $a_1 = 17, a_n = 197, S_n = 2247$	42. $a_1 = -13, a_n = 427, S_n = 18,423$
43. $n = 31, a_n = 78, S_n = 1023$	44. $n = 19, a_n = 103, S_n = 1102$

45. AEROSPACE On the Moon, a falling object falls just 2.65 feet in the first second after being dropped. Each second it falls 5.3 feet farther than in the previous second. How far would an object fall in the first ten seconds after being dropped?

CRITICAL THINKING State whether each statement is *true* or *false*. Explain.

- 46. Doubling each term in an arithmetic series will double the sum.
- **47.** Doubling the number of terms in an arithmetic series, but keeping the first term and common difference the same, will double the sum.
- **48.** WRITING IN MATH Answer the question that was posed at the beginning of the lesson.

How do arithmetic series apply to amphitheaters?

Include the following in your answer:

- explanations of what the sequence and the series that can be formed from the given numbers represent, and
- two ways to find the amphitheater capacity if it has ten rows of seats.



49.	18 + 22 + 26 + 30 + 30	+50 = ?		
	A 146	B 272	C 306	D 340
50.	The angles of a triar angle measures 36°,	gle form an arithme what is the measure	etic sequence of the large	e. If the smallest est angle?
	A 60°		B 72°	
	C 84°		D 144°	36°
Use	e a graphing calculat	or to find the sum or 50^{50}	of each arith	metic series.
51.	$\sum_{n=21}^{\infty} (2n+5)$	52. $\sum_{n=10}^{\infty} (3n-1)$		53. $\sum_{n=20}^{\infty} (4n+3)$

Maintain Your Skills

Graphing Calculator

Mixed Review Find the indicated term of each arithmetic sequence. (Lesson 11-1)

- **54.** $a_1 = 46, d = 5, n = 14$
- **55.** $a_1 = 12, d = -7, n = 22$
- **56. RADIOACTIVITY** The decay of Radon-222 can be modeled by the equation $y = ae^{-0.1813t}$, where *t* is measured in days. What is the half-life of Radon-222? (*Lesson 10-6*)

Solve each equation by completing the square. (Lesson 6-4)

57. $x^2 + 9x + 20.25 = 0$ **58.** $9x^2 + 96x + 256 = 0$ **59.** $x^2 - 3x - 20 = 0$

Simplify. (Lesson 5-6) 60. $5\sqrt{3} - 4\sqrt{3}$ 61. $\sqrt{26} \cdot \sqrt{39} \cdot \sqrt{14}$ 62. $(\sqrt{10} - \sqrt{6})(\sqrt{5} + \sqrt{3})$

Getting Ready for PREREQUISITE SKILL Evaluate the expression $a \cdot b^{n-1}$ for the given values of the Next Lesson a, b, and n. (To review evaluating expressions, see Lesson 1-1.)

63. a = 1, b = 2, n = 5 **64.** a = 2, b = -3, n = 4 **65.** $a = 18, b = \frac{1}{3}, n = 6$

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Lesson 11-2 Arithmetic Series 587

11-3 Geometric Sequences

do geometric sequences apply

What You'll Learn

- Use geometric sequences.
- Find geometric means.

How

Vocabulary

- geometric sequence
- common ratio
- geometric means

If you have ever bounced a ball, you know that when you drop it, it never rebounds to the height from which you dropped it. Suppose a ball is dropped from a height of three feet, and each time it falls, it rebounds to 60% of the height from which it fell. The heights of the ball's rebounds form a sequence.

to a bouncing ball?

GEOMETRIC SEQUENCES The height of the first rebound of the ball is 3(0.6) or 1.8 feet. The height of the second rebound is 1.8(0.6) or 1.08 feet. The height of the third rebound is 1.08(0.6) or 0.648 feet. The sequence of heights, 1.8, 1.08, 0.648, ..., is an example of a **geometric sequence**. A geometric sequence is a sequence in which each term after the first is found by multiplying the previous term by a constant *r* called the **common ratio**.

Height of Rebounds (ft)

Number of Rebounds

As with an arithmetic sequence, you can label the terms of a geometric sequence as a_1, a_2, a_3 , and so on. The *n*th term is a_n and the previous term is a_{n-1} . So, $a_n = r(a_{n-1})$. Thus, $r = \frac{a_n}{a_{n-1}}$. That is, the common ratio can be found by dividing any term by its previous term.

Standardized Example 1) Find the Next Term

Multiple-Choice Test Item

Find the missing te	erm in the geometric	sequence: 8, 20, 50,	125,
A 75	B 200	C 250	D 312.5

Read the Test Item

Since $\frac{20}{8} = 2.5$, $\frac{50}{20} = 2.5$, and $\frac{125}{50} = 2.5$, the sequence has a common ratio of 2.5.

Solve the Test Item

To find the missing term, multiply the last given term by 2.5: 125(2.5) = 312.5.

The answer is D.

You have seen that each term of a geometric sequence can be expressed in terms of r and its previous term. It is also possible to develop a formula that expresses each term of a geometric sequence in terms of r and the first term a_1 . Study the patterns shown in the table on the next page for the sequence 2, 6, 18, 54,



Test Practice

Since the terms of this sequence are increasing, the missing term must be greater than 125. You can immediately eliminate 75

as a possible answer.



Sequence	numbers	2	6	18	54	
	symbols	a ₁	a ₂	a ₃	a ₄	 a _n
Expressed in	numbers	2	2(3)	6(3)	18(3)	
the Previous Term	symbols	a ₁	a₁ · r	a₂ · r	a ₃ · r	 $a_{n-1} \cdot r$
Everyoood in		2	2(3)	2(9)	2(27)	
Terms of <i>r</i> and	numbers	2(3 ⁰)	2(3 ¹)	2(3 ²)	2(3 ³)	
the First Term	symbols	a ₁ · r ⁰	$a_1 \cdot r^1$	$a_1 \cdot r^2$	$a_1 \cdot r^3$	 $a_1 \cdot r^{n-1}$

The three entries in the last column of the table all describe the nth term of a geometric sequence. This leads us to the following formula for finding the nth term of a geometric sequence.

Key Concept nth Term of a Geometric Sequence

The *n*th term a_n of a geometric sequence with first term a_1 and common ratio *r* is given by $a_n = a_1 \cdot r^{n-1}$,

where *n* is any positive integer.

Example 2) Find a Particular Term

Find the eighth term of a geometric sequence for which $a_1 = -3$ and r = -2. $a_n = a_1 \cdot r^{n-1}$ Formula for *n*th term $a_8 = (-3) \cdot (-2)^{8-1}$ $n = 8, a_1 = -3, r = -2$ $a_8 = (-3) \cdot (-128)$ $(-2)^7 = -128$ $a_8 = 384$ Multiply. The eighth term is 384.

Example 3 Write an Equation for the nth Term

Write an equation for the *n*th term of the geometric sequence 3, 12, 48, 192, In this sequence, $a_1 = 3$ and r = 4. Use the *n*th term formula to write an equation. $a_n = a_1 \cdot r^{n-1}$ Formula for *n*th term $a_n = 3 \cdot 4^{n-1}$ $a_1 = 3, r = 4$ An equation is $a_n = 3 \cdot 4^{n-1}$.

You can also use the formula for the *n*th term if you know the common ratio and one term of a geometric sequence, but not the first term.

Example 4 Find a Term Given the Fourth Term and the Ratio

Find the tenth term of a geometric sequence for which $a_4 = 108$ and r = 3.

First, find the value of a_1 .Now find a_{10} . $a_n = a_1 \cdot r^{n-1}$ Formula for *n*th term $a_4 = a_1 \cdot 3^{4-1}$ n = 4, r = 3 $108 = 27a_1$ $a_4 = 108$ $4 = a_1$ Divide each side by 27.Now find a_{10} .Now find a_{10} . $a_n = a_1 \cdot r^{n-1}$ Formula for *n*th term $a_1 = 4, r = 3$ $a_1 = 78,732$ Use a calculator.The tenth term is 78,732.

.....

www.algebra2.com/extra_examples



GEOMETRIC MEANS In Lesson 11-1, you learned that missing terms between two nonsuccessive terms in an arithmetic sequence are called *arithmetic means*. Similarly, the missing terms(s) between two nonsuccessive terms of a geometric sequence are called **geometric means**. For example, 6, 18, and 54 are three geometric means between 2 and 162 in the sequence 2, 6, 18, 54, 162, ... You can use the common ratio to find the geometric means in a given sequence.

Example 5 Find Geometric Means

Find three geometric means between 2.25 and 576.

Use the *n*th term formula to find the value of *r*. In the sequence 2.25, _?_, _?_, _?_, _?_, 576, a_1 is 2.25 and a_5 is 576.

 $a_n = a_1 \cdot r^{n-1}$ Formula for nth term $a_5 = 2.25 \cdot r^{5-1}$ $n = 5, a_1 = 2.25$ $576 = 2.25r^4$ $a_5 = 576$ $256 = r^4$ Divide each side by 2.25. $\pm 4 = r$ Take the fourth root of each side.

There are two possible common ratios, so there are two possible sets of geometric means. Use each value of r to find three geometric means.

r = 4	r = -4
$a_2 = 2.25(4)$ or 9	$a_2 = 2.25(-4)$ or -9
$a_3 = 9(4)$ or 36	$a_3 = -9(-4)$ or 36
$a_4 = 36(4) \text{ or } 144$	$a_4 = 36(-4)$ or -144
The geometric means are 9, 36, and 144,	or -9, 36, and -144.

Check for Understanding

Concept Check 1. Decide whether each sequence is *arithmetic* or *geometric*. Explain.

a. 1, -2, 4, -8, ... **b.** 1, -2, -5, -8, ...

- 2. **OPEN ENDED** Write a geometric sequence with a common ratio of $\frac{2}{3}$.
- **3. FIND THE ERROR** Marika and Lori are finding the seventh term of the geometric sequence 9, 3, 1,

MarikaLori
$$r = \frac{3}{9} \text{ or } \frac{1}{3}$$
 $r = \frac{9}{3} \text{ or } 3$ $a_7 = 9(\frac{1}{3})^{7-1}$ $a_7 = 9 \cdot 3^{7-1}$ $= \frac{1}{81}$ $= 6561$

Who is correct? Explain your reasoning.

Guided Practice Find the next two terms of each geometric sequence.

4. 20, 30, 45, ...

5.
$$-\frac{1}{4}, \frac{1}{2}, -1, \dots$$

6. Find the first five terms of the geometric sequence for which $a_1 = -2$ and r = 3.

Study Tip

Alternate Method You may prefer this method. The three means will be 2.25*r*, 2.25*r*², and 2.25*r*³. Then the common ratio is $r = \frac{576}{2.25r^3}$ or $r^4 = \frac{576}{2.25}$. Thus, r = 4.



7. Find a_9 for the geometric sequence 60, 30, 15,

Find the indicated term of each geometric sequence.

8.
$$a_1 = 7, r = 2, n = 4$$

10. Write an equation for the *n*th term of the geometric sequence 4, 8, 16,

9. $a_3 = 32, r = -0.5, n = 6$

11. Find two geometric means between 1 and 27.



Practice and Apply

Homework Help

For Exercises	See Examples
13-24	1
25–30, 33– 38, 47, 48	2
31, 32	4
39-42	3
43-46	5

Extra Practice

See page 852.

----F-0-----

More About. . .



Art •-----

The largest ever ice construction was an ice palace built for a carnival in St. Paul, Minnesota, in 1992. It contained 10.8 million pounds of ice. **Source:** The Guinness Book of Records Find the next two terms of each geometric sequence.

13.	405, 135, 45,	14.	81, 108, 144,
15.	16, 24, 36,	16.	162, 108, 72,
17.	$\frac{5}{2}, \frac{5}{3}, \frac{10}{9}, \dots$	18.	$\frac{1}{3}, \frac{5}{6}, \frac{25}{12}, \dots$
19.	1.25, -1.5, 1.8,	20.	1.4, -3.5, 8.75,

Find the first five terms of each geometric sequence described.

	0	-
21. $a_1 = 2, r = -3$		22. $a_1 = 1, r = 4$
23. $a_1 = 243, r = \frac{1}{3}$		24. $a_1 = 576, r = -\frac{1}{2}$
25. Find a_7 if $a_n = 12 \left(\frac{1}{2}\right)^{n-1}$.		
26. If $a_n = \frac{1}{2} \cdot 6^{n-1}$, what is a_6 ?		

Find the indicated term of each geometric sequence.

27. $a_1 = \frac{1}{3}, r = 3, n = 8$	28. $a_1 = \frac{1}{64}, r = 4, n = 9$
29. $a_1 = 16,807, r = \frac{3}{7}, n = 6$	30. $a_1 = 4096, r = \frac{1}{4}, n = 8$
31. $a_4 = 16, r = 0.5, n = 8$	32. $a_6 = 3, r = 2, n = 12$
33. a_9 for $\frac{1}{5}$, 1, 5,	34. a_7 for $\frac{1}{32}$, $\frac{1}{16}$, $\frac{1}{8}$,
35. <i>a</i> ₈ for 4, -12, 36,	36. <i>a</i> ₆ for 540, 90, 15, …

- **37. ART** A one-ton ice sculpture is melting so that it loses one-fifth of its weight per hour. How much of the sculpture will be left after five hours? Write the answer in pounds.
- **38. SALARIES** Geraldo's current salary is \$40,000 per year. His annual pay raise is always a percent of his salary at the time. What would his salary be if he got four consecutive 4% increases?

40. 64, 16, 4, ...

42. 4, -12, 36, ...

Write an equation for the *n*th term of each geometric sequence.

39.	36, 12, 4,
41.	-2, 10, -50,

Find the geometric means in each sequence.

43.	9, _	?_,	<u>?</u> ,	_?_,1	44	
45.	32,	_?_,	?,	_?_,	_?_,1	L

44. 4, <u>?</u>, <u>?</u>, <u>?</u>, 324 **46.** 3, <u>?</u>, <u>?</u>, <u>?</u>, <u>?</u>, 96

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Lesson 11-3 Geometric Sequences 591

MEDICINE For Exercises 47 and 48, use the following information.

Iodine-131 is a radioactive element used to study the thyroid gland.

- 47. **RESEARCH** Use the Internet or other resource to find the *half-life* of Iodine-131, rounded to the nearest day. This is the amount of time it takes for half of a sample of Iodine-131 to decay into another element.
- **48.** How much of an 80-milligram sample of Iodine-131 would be left after 32 days?

CRITICAL THINKING Determine whether each statement is *true* or *false*. If true, explain. If false, provide a counterexample.

- **49.** Every sequence is either arithmetic or geometric.
- 50. There is no sequence that is both arithmetic and geometric.
- 51. WRITING IN MATH Answer the question that was posed at the beginning of the lesson.

How do geometric sequences apply to a bouncing ball?

Include the following in your answer:

- the first five terms of the sequence of heights from which the ball falls, and
- any similarities or differences in the sequences for the heights the ball rebounds and the heights from which the ball falls.



52.	Find the missing ter	rm in the geometric s	sequence: -5, 10, -2	0,40,
	▲ -80	B -35	(C) 80	D 100
53.	What is the tenth te	rm in the geometric	sequence: 144, 72, 36	, 18,?
	A 0	B $\frac{9}{64}$	$\bigcirc \frac{9}{32}$	(D) $\frac{9}{16}$

Maintain Your Skills

Mixed Review Find S_n for each arithmetic series described. (Lesson 11-2) **55.** $a_1 = -5, d = 3, n = 14$ **54.** $a_1 = 11, a_n = 44, n = 23$ Find the arithmetic means in each sequence. (Lesson 11-1) **57.** -8, <u>?</u>, <u>?</u>, -24 **56.** 15, _?_, _?_, 27

> **58. GEOMETRY** Find the perimeter of a triangle with vertices at (2, 4), (-1, 3)and (1, -3). (Lesson 8-1)

Getting Ready for PREREQUISITE SKILL Evaluate each expression. (To review expressions, see Lesson 1-1.) the Next Lesson **61.** $\frac{1 - \left(-\frac{1}{3}\right)^5}{1 - \left(-\frac{1}{3}\right)}$

 $60. \quad \frac{1-\left(\frac{1}{2}\right)^{\circ}}{1-\left(\frac{1}{2}\right)}$ **59.** $\frac{1-2^7}{1-2}$

Practice Quiz 1

Lessons 11-1 through 11-3

Find the indicated term of each arithmetic sequence. (Lesson 11-1) **2.** $a_1 = 2, d = \frac{1}{2}, n = 8$ **1.** $a_1 = 7, d = 3, n = 14$ Find the sum of each arithmetic series described. (Lesson 11-2) **3.** $a_1 = 5, a_n = 29, n = 11$ 4. $6 + 12 + 18 + \ldots + 96$ 5. Find a_7 for the geometric sequence 729, -243, 81, (Lesson 11-3)





Graphing Calculator Investigation A Preview of Lesson 11-4

Limits

You may have noticed that in some geometric sequences, the later the term in the sequence, the closer the value is to 0. Another way to describe this is that as *n* increases, a_n approaches 0. The value that the terms of a sequence approach, in this case 0, is called the **limit** of the sequence. Other types of infinite sequences may also have limits. If the terms of a sequence do not approach a unique value, we say that the limit of the sequence does not exist.

Find the limit of the geometric sequence $1, \frac{1}{3}, \frac{1}{9}, \dots$

Step 1 Enter the sequence.

- The formula for this sequence is $a_n = \left(\frac{1}{3}\right)^{n-1}$.
- Position the cursor on L1 in the STAT EDIT Edit ... screen and enter the formula seq(N,N,1,10,1). This generates the values 1, 2, ..., 10 of the index N.
- Position the cursor on L2 and enter the formula seq((1/3)^(N-1),N,1,10,1). This generates the first ten terms of the sequence.
 - **KEYSTROKES:** Review sequences in the Graphing Calculator Investigation on page 585.



Notice that as *n* increases, the terms of the given sequence get closer and closer to 0. If you scroll down, you can see that for $n \ge 8$ the terms are so close to 0 that the calculator expresses them in scientific notation. This suggests that the limit of the sequence is 0.

Step 2 Graph the sequence.

- Use a STAT PLOT to graph the sequence. Use L1 as the Xlist and L2 as the Ylist.
 - **KEYSTROKES:** Review STAT PLOTs on page 87.



[0, 10] scl: 1 by [0, 1] scl: 0.1

The graph also shows that, as *n* increases, the terms approach 0. In fact, for $n \ge 6$, the marks appear to lie on the horizontal axis. This strongly suggests that the limit of the sequence is 0.

Exercises

Use a graphing calculator to find the limit, if it exists, of each sequence.

CONTENTS

2. $a_n = \left(-\frac{1}{2}\right)^n$ **3.** $a_n = 4^n$ **5.** $a_n = \frac{2^n}{2^n + 1}$ **6.** $a_n = \frac{n^2}{n+1}$ **1.** $a_n = \left(\frac{1}{2}\right)^n$ **4.** $a_n = \frac{1}{n^2}$

www.algebra2.com/other_calculator_keystrokes

11-4 Geometric Series

What You'll Learn

- Find sums of geometric series.
- Find specific terms of geometric series.

Vocabulary

geometric series

Iow is e-mailing a joke like a geometric series?

Suppose you e-mail a joke to three friends on Monday. Each of those friends sends the joke on to three of their friends on Tuesday. Each person who receives the joke on Tuesday sends it to three more people on Wednesday, and so on.



GEOMETRIC SERIES Notice that every day, the number of people who read your joke is three times the number that read it the day before. By Sunday, the number of people, including yourself, who have read the joke is 1 + 3 + 9 + 27 + 81 + 243 + 729 + 2187 or 3280!

The numbers 1, 3, 9, 27, 81, 243, 729, and 2187 form a geometric sequence in which $a_1 = 1$ and r = 3. Since 1, 3, 9, 27, 81, 243, 729, 2187 is a geometric sequence, 1 + 3 + 9 + 27 + 81 + 243 + 729 + 2187 is called a **geometric series**. Below are some more examples of geometric sequences and their corresponding geometric series.

Geometric Sequences	Geometric Series
1, 2, 4, 8, 16	1 + 2 + 4 + 8 + 16
4, -12, 36	4 + (-12) + 36
5, 1, $\frac{1}{5}$, $\frac{1}{25}$	$5 + 1 + \frac{1}{5} + \frac{1}{25}$

To develop a formula for the sum of a geometric series, consider the series given in the e-mail situation above.

$$S_{8} = 1 + 3 + 9 + 27 + 81 + 243 + 729 + 2187$$

$$(-) 3S_{8} = 3 + 9 + 27 + 81 + 243 + 729 + 2187 + 6561$$

$$(1 - 3)S_{8} = 1 + 0 + 0 + 0 + 0 + 0 + 0 + 0 - 6561$$
first term in series
$$S_{8} = \frac{1 - 6561}{1 - 3} \text{ or } 3280$$
last term in series multiplied by common ratio; in this case, a_{9}
common ratio

The expression for S_8 can be written as $S_8 = \frac{a_1 - a_1 r^8}{1 - r}$. A rational expression like this can be used to find the sum of any geometric series.

Study Tip

Terms of Geometric Sequences Remember that a_9 can also be written as a_1r^8 .



Key Concept

Sum of a Geometric Series

The sum S_n of the first *n* terms of a geometric series is given by

$$S_n = \frac{a_1 - a_1 r^n}{1 - r}$$
 or $S_n = \frac{a_1 (1 - r^n)}{1 - r}$, where $r \neq 1$.

You cannot use the formula for the sum with a geometric series for which r = 1 because division by 0 would result. In a geometric series with r = 1, the terms are constant. For example, 4 + 4 + 4 + ... + 4 is such a series. In general, the sum of n terms of a geometric series with r = 1 is $n \cdot a^1$.

Example 1) Find the Sum of the First n Terms

• **GENEALOGY** In the book *Roots*, author Alex Haley traced his family history back many generations to the time one of his ancestors was brought to America from Africa. If you could trace your family back for 15 generations, starting with your parents, how many ancestors would there be?

Counting your two parents, four grandparents, eight great-grandparents, and so on gives you a geometric series with $a_1 = 2$, r = 2, and n = 15.

$$S_{n} = \frac{a_{1}(1 - r^{n})}{1 - r}$$
 Sum formula

$$S_{15} = \frac{2(1 - 2^{15})}{1 - 2}$$
 $n = 15, a_{1} = 2, r = 3$

$$S_{15} = 65,534$$
 Use a calculator.

Going back 15 generations, you have 65,534 ancestors.

As with arithmetic series, you can use sigma notation to represent geometric series.

Example 2) Evaluate a Sum Written in Sigma Notation

2

Evaluate $\sum_{n=1}^{6} 5 \cdot 2^{n-1}$.

Method 1

Find the terms by replacing *n* with 1, 2, 3, 4, 5, and 6. Then add.

$$\sum_{i=1}^{5} 5 \cdot 2^{n-1} = 5(2^{1-1}) + 5(2^{2-1}) + 5(2^{4-1}) + 5(2^{5-1}) + 5(2^{6-1}) + 5(2^{5-1}) + 5(2^{6-1}) = 5(1) + 5(2) + 5(4) + 5(8) + 5(16) + 5(32) = 5 + 10 + 20 + 40 + 80 + 160 = 315$$

Method 2

Since the sum is a geometric series, you can use the formula

$$S_{n} = \frac{a_{1}(1 - r^{n})}{1 - r}.$$

$$S_{6} = \frac{5(1 - 2^{6})}{1 - 2} \quad n = 6, a_{1} = 5, r = 2$$

$$S_{6} = \frac{5(-63)}{-1} \quad 2^{6} = 64$$

$$S_{6} = 315 \quad \text{Simplify.}$$

The sum of the series is 315.

How can you find the sum of a geometric series if you know the first and last terms and the common ratio, but not the number of terms? Remember the formula for the *n*th term of a geometric sequence or series, $a_n = a_1 \cdot r^{n-1}$. You can use this formula to find an expression involving r^n .

$$a_n = a_1 \cdot r^{n-1}$$
 Formula for *n*th term

$$a_n \cdot r = a_1 \cdot r^{n-1} \cdot r$$
 Multiply each side by *r*.

$$a_n \cdot r = a_1 \cdot r^n$$
 $r^{n-1} \cdot r^1 = r^{n-1+1}$ or r^n



Genealogy

When he died in 1992, Samuel Must of Fryburg, Pennsylvania, had a record 824 living descendants. Source: The Guinness Book of Records



www.algebra2.com/extra examples



Lesson 11-4 Geometric Series 595

Now substitute $a_n \cdot r$ for $a_1 \cdot r^n$ in the formula for the sum of a geometric series. The result is $S_n = \frac{a_1 - a_n r}{1 - r}$.

Example 3 Use the Alternate Formula for a Sum

Find the sum of a geometric series for which $a_1 = 15,625$, $a_n = -5$, and $r = -\frac{1}{5}$. Since you do not know the value of *n*, use the formula derived above.

 $S_{n} = \frac{a_{1} - a_{n}r}{1 - r}$ Alternate sum formula $= \frac{15,625 - (-5)(-\frac{1}{5})}{1 - (-\frac{1}{5})} \qquad a_{1} = 15,625, a_{n} = -5, r = -\frac{1}{5}$ $= \frac{15,624}{\frac{6}{5}} \text{ or } 13,020 \qquad \text{Simplify.}$

SPECIFIC TERMS You can use the formula for the sum of a geometric series to help find a particular term of the series.

Example 4 Find the First Term of a Series Find a_1 in a geometric series for which $S_8 = 39,360$ and r = 3.

$$S_{n} = \frac{a_{1}(1 - r^{n})}{1 - r}$$
Sum formula

$$39,360 = \frac{a_{1}(1 - 3^{8})}{1 - 3}$$
$$S_{8} = 39,360; r = 3; n = 8$$

$$39,360 = \frac{-6560a_{1}}{-2}$$
Subtract.

$$39,360 = 3280a_{1}$$
Divide.

$$12 = a_{1}$$
Divide each side by 3280.

The first term of the series is 12.

Check for Understanding

Concept Check 1. OPEN ENDED Write a geometric series for which $r = \frac{1}{2}$ and n = 4.

- **2.** Explain, using geometric series, why the polynomial $1 + x + x^2 + x^3$ can be written as $\frac{x^4 1}{x 1}$, assuming $x \neq 1$.
- **3.** Explain how to write the series 2 + 12 + 72 + 432 + 2592 using sigma notation.

Guided Practice Find S_n for each geometric series described.

1

4.
$$a_1 = 12, a_5 = 972, r = -3$$
5. $a_1 = 3, a_n = 46,875, r = -5$ **6.** $a_1 = 5, r = 2, n = 14$ **7.** $a_1 = 243, r = -\frac{2}{3}, n = 5$

Find the sum of each geometric series.

8. $54 + 36 + 24 + 16 + \dots$ to 6 terms **9.** $3 - 6 + 12 - \dots$ to 7 terms

CONTENTS

10.
$$\sum_{n=1}^{5} \frac{1}{4} \cdot 2^{n-1}$$

9.
$$3 - 6 + 12 - \dots$$
 to 7 t
11. $\sum_{n=1}^{7} 81 \left(\frac{1}{3}\right)^{n-1}$

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Find the indicated term for each geometric series described.

12. $S_n = \frac{381}{64}, r = \frac{1}{2}, n = 7; a_1$

13. $S_n = 33, a_n = 48, r = -2; a_1$

Application 14. WEATHER Heavy rain caused a river to rise. The river rose three inches the first day, and each additional day it rose twice as much as the previous day. How much did the river rise in five days?

Practice and Apply

Homework Help

See Examples
1, 3
2
4

Extra Practice

See page 852.



Legends •·····

Some of the best-known legends involving a king are the Arthurian legends. According to legend, King Arthur reigned over Britain before the Saxon conquest. Camelot was the most famous castle in the medieval legends of King Arthur.

Find S_n for each geometric series describe	ed.
15. $a_1 = 2, a_6 = 486, r = 3$	16. $a_1 = 3, a_8 = 384, r = 2$
17. $a_1 = 1296, a_n = 1, r = -\frac{1}{6}$	18. $a_1 = 343, a_n = -1, r = -\frac{1}{7}$
19. $a_1 = 4, r = -3, n = 5$	20. $a_1 = 5, r = 3, n = 12$
21. $a_1 = 2401, r = -\frac{1}{7}, n = 5$	22. $a_1 = 625, r = \frac{3}{5}, n = 5$
23. $a_1 = 162, r = \frac{1}{3}, n = 6$	24. $a_1 = 80, r = -\frac{1}{2}, n = 7$
25. $a_1 = 625, r = 0.4, n = 8$	26. $a_1 = 4, r = 0.5, n = 8$
27. $a_2 = -36$, $a_5 = 972$, $n = 7$	28. $a_3 = -36, a_6 = -972, n = 10$

- **29. HEALTH** Contagious diseases can spread very quickly. Suppose five people are ill during the first week of an epidemic and that each person who is ill spreads the disease to four people by the end of the next week. By the end of the tenth week of the epidemic, how many people have been affected by the illness?
- **30. LEGENDS** There is a legend of a king who wanted to reward a boy for a good deed. The king gave the boy a choice. He could have \$1,000,000 at once, or he could be rewarded daily for a 30-day month, with one penny on the first day, two pennies on the second day, and so on, receiving twice as many pennies each day as the previous day. How much would the second option be worth?

Find the sum of each geometric series.

31.	$4096 - 512 + 64 - \dots tetter = 1000$	5 5 te	erms	32.	7 + 21 +	63	+ to 10 terms
33.	$\frac{1}{16} + \frac{1}{4} + 1 + \dots$ to 7 te	rms		34.	$\frac{1}{9} - \frac{1}{3} +$	1 –	to 6 terms
35.	$\sum_{n=1}^{9} 5 \cdot 2^{n-1}$	36.	$\sum_{n=1}^{6} 2(-3)^{n-1}$	1		37.	$\sum_{n=1}^{7} 144 \left(-\frac{1}{2}\right)^{n-1}$
38.	$\sum_{n=1}^{8} 64 \left(\frac{3}{4}\right)^{n-1}$	39.	$\sum_{n=1}^{20} 3 \cdot 2^{n-2}$	1		40.	$\sum_{n=1}^{16} 4 \cdot 3^{n-1}$

Find the indicated term for each geometric series described.

41. $S_n = 165, a_n = 48, r = -\frac{2}{3}; a_1$	42. $S_n = 688, a_n = 16, r = -\frac{1}{2}; a_1$
43. $S_n = -364, r = -3, n = 6; a_1$	44. $S_n = 1530, r = 2, n = 8; a_1$
45. $S_n = 315, r = 0.5, n = 6; a_2$	46. $S_n = 249.92, r = 0.2, n = 5, a_3$

47. LANDSCAPING Rob is helping his dad install a fence. He is using a sledgehammer to drive the pointed fence posts into the ground. On his first swing, he drives a post five inches into the ground. Since the soil is denser the deeper he drives, on each swing after the first, he can only drive the post 30% as far into the ground as he did on the previous swing. How far has he driven the post into the ground after five swings?



48. CRITICAL THINKING If a_1 and r are integers, explain why the value of

 $\frac{a_1 - a_1 r^n}{1 - r}$ must also be an integer.

49. WRITING IN MATH Answer the question that was posed at the beginning of the lesson.

How is e-mailing a joke like a geometric series?

Include the following in your answer:

- how the related geometric series would change if each person e-mailed the joke on to four people instead of three, and
- how the situation could be changed to make it better to use a formula than to add terms.



50. The first term of a geometric series is -1, and the common ratio is -3. How many terms are in the series if its sum is 182?

|--|

51. What is the first term in a geometric series with ten terms, a common ratio of 0.5, and a sum of 511.5?



Use a graphing calculator to find the sum of each geometric series.

52. $\sum_{n=1}^{20} 3(-2)^{n-1}$

53. $\sum_{n=1}^{15} 2\left(\frac{1}{2}\right)^{n-1}$

Maintain Your Skills

Mixed Review Find the geometric means in each sequence. (Lesson 11-3) **56.** -2, <u>?</u>, <u>?</u>, <u>?</u>, <u>?</u>,

Find the sum of each arithmetic series. (Lesson 11-2)

```
57. 50 + 44 + 38 + ... + 8
```

58. $\sum_{n=1}^{12} (2n+3)$

54. $\sum_{n=1}^{10} 5(0.2)^{n-1}$

ENTERTAINMENT For Exercises 59–61, use the table that shows the number of drive-in movie screens in the United States for 1995–2000. (Lesson 2-5)

1111						
Near	1 995	1996	1.			1111
100 ms	848	826	1997	1998	1 999	2000
Screen			815	750	737	637
Source: Nation	al Association	of Theatre Ow	iners			TTTT I

- **59.** Draw a scatter plot, in which *x* is the number of years since 1995.
- **60.** Find a prediction equation.
- 61. Predict the number of screens in 2010.

Online Research Data Update For the latest statistics on the movie industry, visit: www.algebra2.com/data_update

Getting Ready for PREREQUISITE SKILL Evaluate $\frac{a}{1-b}$ for the given values of a and b. the Next Lesson (To review evaluating expressions, see Lesson 1-1.) **64.** $a = \frac{1}{3}, b = -\frac{1}{3}$ **63.** a = 3 $b = -\frac{1}{2}$ 62 a - 1 b - 1

65.
$$a = \frac{1}{2}, b = \frac{1}{4}$$

66. $a = -1, b = 0.5$
67. $a = 0.9, b = -0.5$



11-5 Infinite Geometric Series

What You'll Learn

- Find the sum of an infinite geometric series.
- Write repeating decimals as fractions.

Vocabulary

infinite geometric seriespartial sum

How does an infinite geometric series apply to a bouncing ball?

Refer to the beginning of Lesson 11-3. Suppose you wrote a geometric series to find the sum of the heights of the rebounds of the ball. The series would have no last term because theoretically there is no last bounce of the ball. For every rebound of the ball, there is another rebound, 60% as high. Such a geometric series is called an **infinite geometric series**.



INFINITE GEOMETRIC SERIES Consider the infinite geometric series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$. You have already learned how to find the sum S_n of the first *n* terms of a geometric series. For an infinite series, S_n is called a **partial sum** of the series. The table and graph show some values of S_n .

n	S _n	S _n 1 .0
1	$\frac{1}{2}$ or 0.5	0.9
2	$\frac{3}{4}$ or 0.75	E 0.7
3	7/8 or 0.875	6 0.5 5 0.4 7 0.4
4	15/16 or 0.9375	0.3
5	$\frac{31}{32}$ or 0.96875	0 1 2 3 4 5 6 7
6	63/64 or 0.984375	Term
7	$\frac{127}{128}$ or 0.9921875	

Study Tip

Absolute Value Recall that |r| < 1means -1 < r < 1.

Notice that as *n* increases, the partial sums level off and approach a limit of 1. This leveling-off behavior is characteristic of infinite geometric series for which |r| < 1.



Let's look at the formula for the sum of a finite geometric series and use it to find a formula for the sum of an infinite geometric series.

$$S_n = \frac{a_1 - a_1 r^n}{1 - r}$$
 Sum of first *n* terms
$$= \frac{a_1}{1 - r} - \frac{a_1 r^n}{1 - r}$$
 Write the fraction as a difference of fractions.

If -1 < r < 1, the value of r^n will approach 0 as n increases. Therefore, the partial sums of an infinite geometric series will approach $\frac{a_1}{1-r} - \frac{a_1(0)}{1-r}$ or $\frac{a_1}{1-r}$. This expression gives the sum of an infinite geometric series.

Key Concept	Sum of an Infinite Geometric Series				
The sum S of an infinite geometric series with $-1 < r < 1$ is given by					
$S = \frac{a_1}{a_1}$					

Study Tip

Formula for Sum if -1 < r < 1To convince yourself of this formula, make a table of the first ten partial sums of the geometric series with $r = \frac{1}{2}$ and $a_1 = 100$.

Term Number	Term	Partial Sum
1	100	100
2	50	150
3	25	175
:	:	:
10		

Complete the table and compare the sum that the series is approaching to that obtained by using the formula.

An infinite geometric series for which $|r| \ge 1$ does not have a sum. Consider the series 1 + 3 + 9 + 27 + 81 + In this series, $a_1 = 1$ and r = 3. The table shows some of the partial sums of this series. As *n* increases, S_n rapidly increases and has no limit. That is, the partial sums do not approach a particular value.

n	S _n
5	121
10	29,524
15	7,174,453
20	1,743,392,200

Example 1 Sum of an Infinite Geometric Series

Find the sum of each infinite geometric series, if it exists.

a.
$$\frac{1}{2} + \frac{3}{8} + \frac{9}{32} + \dots$$

First, find the value of *r* to determine if the sum exists.

$$a_1 = \frac{1}{2}$$
 and $a_2 = \frac{3}{8}$, so $r = \frac{\frac{3}{8}}{\frac{1}{2}}$ or $\frac{3}{4}$. Since $\left|\frac{3}{4}\right| < 1$, the sum exists

Now use the formula for the sum of an infinite geometric series.

$$S = \frac{a_{1}}{1 - r}$$
 Sum formula
= $\frac{\frac{1}{2}}{1 - \frac{3}{4}}$ $a_{1} = \frac{1}{2}r + \frac{3}{4}$
= $\frac{\frac{1}{2}}{\frac{1}{4}}$ or 2 Simplify.

The sum of the series is 2.

b.
$$1 - 2 + 4 - 8 + \dots$$

 $a_1 = 1$ and $a_2 = -2$, so $r = \frac{-2}{1}$ or -2 . Since $|-2| \ge 1$, the sum does not exist.



In Lessons 11-2 and 11-4, we used sigma notation to represent finite series. You can also use sigma notation to represent infinite series. An *infinity symbol* ∞ is placed above the Σ to indicate that a series is infinite.

Example 2 Infinite Series in Sigma Notation Evaluate $\sum_{n=1}^{\infty} 24\left(-\frac{1}{5}\right)^{n-1}$. In this infinite geometric series, $a_1 = 24$ and $r = -\frac{1}{5}$. $S = \frac{a_1}{1-r}$ Sum formula $= \frac{24}{1-\left(-\frac{1}{5}\right)}$ $a_1 = 24, r = -\frac{1}{5}$ $= \frac{24}{\frac{6}{5}}$ or 20 Simplify. Thus, $\sum_{n=1}^{\infty} 24\left(-\frac{1}{5}\right)^{n-1} = 20$.

REPEATING DECIMALS The formula for the sum of an infinite geometric series can be used to write a repeating decimal as a fraction. Remember that decimals with bar notation such as 0.2 and 0.47 represent 0.222222... and 0.474747..., respectively. Each of these expressions can be written as an infinite geometric series.

Example 3 Write a Repeating Decimal as a Fraction

Write $0.\overline{39}$ as a fraction.

Method 1 Method 2 $S = 0.\overline{39}$ Write the repeating decimal as a sum. Label the given decimal. $0.\overline{39} = 0.393939...$ *S* = 0.393939... **Repeating decimal** $= 0.39 + 0.0039 + 0.000039 + \dots$ 100S = 39.393939...Multiply each side by $=\frac{39}{100}+\frac{39}{10\,000}+\frac{39}{1\,000\,000}+\dots$ 100. 99S = 39Subtract the second In this series, $a_1 = \frac{39}{100}$ and $r = \frac{1}{100}$. equation from the third. $S = \frac{a_1}{1-r}$ Sum formula $S = \frac{39}{99} \text{ or } \frac{13}{33}$ Divide each side by 99. $=\frac{\frac{39}{100}}{1-\frac{1}{100}} \quad a_1 = \frac{39}{100}, r = \frac{1}{100}$ $=\frac{\overline{100}}{99}$ Subtract. $=\frac{39}{99} \text{ or } \frac{13}{33}$ Simplify. Thus, $0.\overline{39} = \frac{13}{33}$.



Lesson 11-5 Infinite Geometric Series 601

Check for Understanding

Concept Check 1. OPEN ENDED Write the series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$ using sigma notation.

- **2.** Explain why 0.999999... = 1.
- 3. FIND THE ERROR Miguel and Beth are discussing the series $-\frac{1}{3} + \frac{4}{9} \frac{16}{27} + \dots$ Miguel says that the sum of the series is $-\frac{1}{7}$. Beth says that the series does not have a sum. Who is correct? Explain your reasoning.



Guided Practice Find the sum of each infinite geometric series, if it exists.

4. $a_1 = 36, r = \frac{2}{3}$	5. $a_1 = 18, r = -1.5$
6. 16 + 24 + 36 +	7. $\frac{1}{4} + \frac{1}{6} + \frac{2}{18} + \dots$
8. $6 - 2.4 + 0.96 - \dots$	9. $\sum_{n=1}^{\infty} 40 \left(\frac{3}{5}\right)^{n-1}$
Write each repeating decimal as a fraction	,

	lie each	repeating decimal as a machon.		
10.	0.5	11. 0.73	12.	$0.\overline{175}$

Application 13. CLOCKS Jasmine's old grandfather clock is broken. When she tries to set the pendulum in motion by holding it against the side of the clock and letting it go, it first swings 24 centimeters to the other side, then 18 centimeters back, then 13.5 centimeters, and so on. What is the total distance that the pendulum swings?

Practice and Apply

Homework Help

For Exercises	See Examples
14-27,	1
32-39 28-31	2
40-47	3

Extra Practice

See page 852.

Find the sum of each infinite geometric series, if it exists.

14. $a_1 = 4, r = \frac{5}{7}$	15. $a_1 = 14, r = \frac{7}{3}$	16. $a_1 = 12, r = -0.6$
17. $a_1 = 18, r = 0.6$	18. 16 + 12 + 9 +	19. -8 - 4 - 2
20. 12 - 18 + 24	21. 18 - 12 + 8	22. $1 + \frac{2}{3} + \frac{4}{9} + \dots$
23. $\frac{5}{3} + \frac{25}{3} + \frac{125}{3} + \dots$	24. $\frac{5}{3} - \frac{10}{9} + \frac{20}{27} - \dots$	25. $\frac{3}{2} - \frac{3}{4} + \frac{3}{8} - \dots$
26. 3 + 1.8 + 1.08 +	27. 1 - 0.5 + 0.25	28. $\sum_{n=1}^{\infty} 48 \left(\frac{2}{3}\right)^{n-1}$
29. $\sum_{n=1}^{\infty} \left(\frac{3}{8}\right) \left(\frac{3}{4}\right)^{n-1}$	30. $\sum_{n=1}^{\infty} 3(0.5)^{n-1}$	31. $\sum_{n=1}^{\infty} (1.5)(0.25)^{n-1}$

32. CHILD'S PLAY Kimimela's little sister likes to swing at the playground. Yesterday, Kimimela pulled the swing back and let it go. The swing traveled a distance of 9 feet before heading back the other way. Each swing afterward was only 70% as long as the previous one. Find the total distance the swing traveled.



GEOMETRY For Exercises 33 and 34, refer to square *ABCD*, which has a perimeter of 40 centimeters.

If the midpoints of the sides are connected, a smaller square results. Suppose the process of connecting midpoints of sides and drawing new squares is continued indefinitely.



- **33.** Write an infinite geometric series to represent the sum of the perimeters of all of the squares.
- 34. Find the sum of the perimeters of all of the squares.
- **35. AVIATION** A hot-air balloon rises 90 feet in its first minute of flight. In each succeeding minute, it rises only 90% as far as it did during the preceding minute. What is the final height of the balloon?
- **36.** The sum of an infinite geometric series is 81, and its common ratio is $\frac{2}{3}$. Find the first three terms of the series.
- **37.** The sum of an infinite geometric series is 125, and the value of *r* is 0.4. Find the first three terms of the series.
- **38.** The common ratio of an infinite geometric series is $\frac{11}{16}$, and its sum is $76\frac{4}{5}$. Find the first four terms of the series.
- **39.** The first term of an infinite geometric series is -8, and its sum is $-13\frac{1}{3}$. Find the first four terms of the series.

Write each repeating decimal as a fraction.

40. 0.7	41. 0.1	42. 0. 36	43. 0. 82
44. 0. <u>246</u>	45. 0.427	46. 0.45	47. 0.231

- **48. CRITICAL THINKING** Derive the formula for the sum of an infinite geometric series by using the technique in Lessons 11-2 and 11-4. That is, write an equation for the sum *S* of a general infinite geometric series, multiply each side of the equation by *r*, and subtract equations.
- **49.** WRITING IN MATH Answer the question that was posed at the beginning of the lesson.

How does an infinite geometric series apply to a bouncing ball?

Include the following in your answer:

- some formulas you might expect to see on the chalkboard if the character in the comic strip really was discussing a bouncing ball, and
- an explanation of how to find the total distance traveled, both up and down, by the bouncing ball described at the beginning of Lesson 11-3.

Standardized 50. Wh Test Practice con

More About.

Aviation ⊷

The largest hot-air balloon ever flown had a capacity

of 2.6 million cubic feet.

Source: The Guinness Book of Records

50. What is the sum of an infinite geometric series with a first term of 6 and a common ratio of $\frac{1}{2}$?

(A) 3 (B) 4 (C) 9
51.
$$2 + \frac{2}{3} + \frac{2}{9} + \frac{2}{27} + ... =$$

(A) $\frac{3}{2}$ (B) $\frac{80}{27}$ (C) 3

CONTENTS

D does not exist

www.algebra2.com/self_check_quiz

Lesson 11-5 Infinite Geometric Series 603

D 12

Maintain Your Skills

Mixed Review Find S, for each geometric series described. (Lesson 11-4)

52.
$$a_1 = 1, a_6 = -243, r = -3$$

53. $a_1 = 72, r = \frac{1}{3}, n = 7$

54. PHYSICS A vacuum pump removes 20% of the air from a container with each stroke of its piston. What percent of the original air remains after five strokes of the piston? *(Lesson 11-3)*

 Solve each equation or inequality. Check your solution. (Lesson 10-1)

 55. $6^x = 216$ 56. $2^{2x} = \frac{1}{8}$ 57. $3^{x-2} \ge 27$

Simplify each expression. (Lesson 9-2) 58. $\frac{-2}{ab} + \frac{5}{a^2}$ 59. $\frac{1}{x-3} - \frac{2}{x+1}$ 60. $\frac{1}{x^2+6x+8} + \frac{3}{x+4}$

Write an equation for the circle that satisfies each set of conditions. (Lesson 8-3)

- **61.** center (2, 4), radius 6
- **62.** endpoints of a diameter at (7, 3) and (-1, -5)

Find all the zeros of each function. (Lesson 7-5) 63. $f(x) = 8x^3 - 36x^2 + 22x + 21$ 64. $g(x) = 12x^4 + 4x^3 - 3x^2 - x$

Write a quadratic equation with the given roots. Write the equation in the form $ax^2 + bx + c = 0$, where *a*, *b*, and *c* are integers. (Lesson 6-3)

RECREATION For Exercises 68 and 69, refer to the graph at the right. (*Lesson 2-3*)

- **68.** Find the average rate of change of the number of visitors to Yosemite National Park from 1996 to 1999.
- **69.** Was the number of visitors increasing or decreasing from 1996 to 1999?







Spreadsheet Investigation

A Preview of Lesson 11-6

Amortizing Loans

When a payment is made on a loan, part of the payment is used to cover the interest that has accumulated since the last payment. The rest is used to reduce the *principal*, or original amount of the loan. This process is called *amortization*. You can use a spreadsheet to analyze the payments, interest, and balance on a loan. A table that shows this kind of information is called an *amortization schedule*.

Example

Marisela just bought a new sofa for \$495. The store is letting her make monthly payments of \$43.29 at an interest rate of 9% for one year. How much will she still owe after six months?

Every month, the interest on the remaining balance will be $\frac{9\%}{12}$ or 0.75%. You can

find the balance after a payment by multiplying the balance after the previous payment by 1 + 0.0075 or 1.0075 and then subtracting 43.29.

In a spreadsheet, use the column of numbers for the number of payments and use column B for the balance. Enter the interest rate and monthly payment in cells in column A so that they can be easily updated if the information changes.

The spreadsheet at the right shows the formulas for the balances after each of the first six payments. After six months, Marisela still owes \$253.04.

	Loa	ns E	기티
	A	B	-
1	Interest rate	=495*(1+A2)-A5	
2	0.0075	=B1*(1+A2)-A5	
3	- KINGATATATIO	=B2*(1+A2)-A5	
4	Monthly payment	=B3*(1+A2)-A5	
5	43.29	=B4*(1+A2)-A5	
6	00000.0000	=85*(1+A2)-A5	
7		Share and second second	
1	♦ ► ► Sheet1 /	Sh 🛄 🕹 🖡	111

Exercises

- **1.** Let b_n be the balance left on Marisela's loan after *n* months. Write an equation relating b_n and b_{n+1} .
- **2.** Payments at the beginning of a loan go more toward interest than payments at the end. What percent of Marisela's loan remains to be paid after half a year?
- **3.** Extend the spreadsheet to the whole year. What is the balance after 12 payments? Why is it not 0?
- **4.** Suppose Marisela decides to pay \$50 every month. How long would it take her to pay off the loan?
- **5.** Suppose that, based on how much she can afford, Marisela will pay a variable amount each month in addition to the \$43.29. Explain how the flexibility of a spreadsheet can be used to adapt to this situation.
- **6.** Jamie has a three-year, \$12,000 car loan. The annual interest rate is 6%, and his monthly payment is \$365.06. After twelve months, he receives an inheritance which he wants to use to pay off the loan. How much does he owe at that point?

CONTENTS

Recursion and Special Sequences

What You'll Learn

- Recognize and use special sequences.
- Iterate functions.

Vocabulary

11-6

- Fibonacci sequence
- recursive formula
- iteration

is the Fibonacci sequence How illustrated in nature? A shoot on a sneezewort plant must grow for two months before it is strong enough to put out another

shoot. After that, it puts out at least

Month	1	2	3	4	5
Shoots	1	1	2	3	5

one shoot every month.

7 months 6 months 5 months 4 months 3 months 2 months Start

SPECIAL SEQUENCES Notice that the sequence 1, 1, 2, 3, 5, 8, 13, ... has a pattern. Each term in the sequence is the sum of the two previous terms. For example, 8 = 3 + 5 and 13 = 5 + 8. This sequence is called the **Fibonacci sequence**, and it is found in many places in nature.

first term	a_1		1
second term	a_2		1
third term	<i>a</i> ₃	$a_1 + a_2$	1 + 1 = 2
fourth term	a_4	$a_2 + a_3$	1 + 2 = 3
fifth term	a_5	$a_3 + a_4$	2 + 3 = 5
÷	:	:	
<i>n</i> th term	a _n	$a_{n-2} + a_{n-1}$	

The formula $a_n = a_{n-2} + a_{n-1}$ is an example of a **recursive formula**. This means that each term is formulated from one or more previous terms. To be able to use a recursive formula, you must be given the value(s) of the first term(s) so that you can start the sequence and then use the formula to generate the rest of the terms.

Example 1) Use a Recursive Formula

Find the first five terms of the sequence in which $a_1 = 4$ and $a_{n+1} = 3a_n - 2$, $n \geq 1$.

 $a_{n+1} = 3a_n - 2$ Recursive formula $a_{1+1} = 3a_1 - 2$ n = 1 $a_2 = 3(4) - 2 \text{ or } 10 \quad a_1 = 4$ $a_{2+1} = 3a_2 - 2$ n = 2 $a_4 + 1 = 3a_4 - 2$ n = 4

 $a_{3+1} = 3a_3 - 2$ n = 3 $a_4 = 3(28) - 2 \text{ or } 82 \quad a_3 = 28$ $a_3 = 3(10) - 2 \text{ or } 28$ $a_2 = 10$ $a_5 = 3(82) - 2 \text{ or } 244$ $a_4 = 82$

The first five terms of the sequence are 4, 10, 28, 82, and 244.

Reading Math

A recursive formula is often called a recursive relation or a recurrence relation.



Example 2 Find and Use a Recursive Formula

GARDENING Mr. Yazaki discovered that there were 225 dandelions in his garden on the first Saturday of spring. He had time to pull out 100, but by the next Saturday, there were twice as many as he had left. Each Saturday in spring, he removed 100 dandelions, only to find that the number of remaining dandelions had doubled by the following Saturday.

a. Write a recursive formula for the number of dandelions Mr. Yazaki finds in his garden each Saturday.

Let d_n represent the number of dandelions at the beginning of the *n*th Saturday. Mr. Yazaki will pull 100 of these out of his garden, leaving $d_n - 100$. The number d_{n+1} of dandelions the next Saturday will be twice this number. So, $d_{n+1} = 2(d_n - 100)$ or $2d_n - 200$.

b. Find the number of dandelions Mr. Yazaki would find on the fifth Saturday.

On the first Saturday, there were 225 dandelions, so $d_1 = 225$.

 $d_{n+1} = 2d_n - 200$ Recursive formula

On the fifth Saturday, there would be 600 dandelions in Mr. Yazaki's garden.

You can use sequences to analyze some games.

Algebra Activity

Special Sequences

The object of the *Towers of Hanoi* game is to move a stack of n coins from one position to another in the fewest number a_n of moves with these rules.

- You may only move one coin at a time.
- A coin must be placed on top of another coin, not underneath.
- A smaller coin may be placed on top of a larger coin, but not vice versa. For example, a penny may it

CONTENTS



versa. For example, a penny may not be placed on top of a dime.

Model and Analyze

- 1. Draw three circles on a sheet of paper, as shown above. Place a penny on the first circle. What is the least number of moves required to get the penny to the second circle?
- **2.** Place a nickel and a penny on the first circle, with the penny on top. What is the least number of moves that you can make to get the stack to another circle? (Remember, a nickel cannot be placed on top of a penny.)
- **3.** Place a nickel, penny, and dime on the first circle. What is the least number of moves that you can take to get the stack to another circle?

Make a Conjecture

4. Place a quarter, nickel, penny, and dime on the first circle. Experiment to find the least number of moves needed to get the stack to another circle. Make a conjecture about a formula for the minimum number a_n of moves required to move a stack of *n* coins.

www.algebra2.com/extra_examples

Study Tip

Look Back To review composition of functions, see Lesson 7-7. **ITERATION** Iteration is the process of composing a function with itself repeatedly. For example, if you compose a function with itself once, the result is $f \circ f(x)$ or f(f(x)). If you compose a function with itself two times, the result is $f \circ f \circ f(x)$ or f(f(f(x))), and so on.

You can use iteration to recursively generate a sequence. Start with an initial value x_0 . Let $x_1 = f(x_0)$, $x_2 = f(x_1)$ or $f(f(x_0))$, $x_3 = f(x_2)$ or $f(f(f(x_0)))$, and so on.

Example 3 Iterate a Function

Find the first three iterates $x_{1'} x_{2'} x_3$ of the function f(x) = 2x + 3 for an initial value of $x_0 = 1$.

To find the first iterate x_1 , find the value of the function for $x_0 = 1$.

 $x_1 = f(x_0)$ Iterate the function.

$$= f(1)$$
 $x_0 =$

= 2(1) + 3 or 5 Simplify.

To find the second iterate x_2 , substitute x_1 for x.

1

 $x_2 = f(x_1)$ Iterate the function.

= f(5) $x_1 = 5$

= 2(5) + 3 or 13 Simplify.

Substitute x_2 for x to find the third iterate.

 $x_3 = f(x_2)$ Iterate the function. = f(13) $x_2 = 13$ = 2(13) + 3 or 29 Simplify.

Therefore, 1, 5, 13, 29 is an example of a sequence generated using iteration.

.....

Check for Understanding

Concept Check	1. Write recursive formulas for the <i>n</i> th terms of arithmetic and geometric sequences.	
	2. OPEN ENDED Write a recursive formula for a sequence whose first three to are 1, 1, and 3.	erms
	3. State whether the statement $x_n \neq x_{n-1}$ is <i>sometimes, always,</i> or <i>never</i> true if $x_n = f(x_{n-1})$. Explain.	
Guided Practice	Find the first five terms of each sequence.	
	4. $a_1 = 12, a_{n+1} = a_n - 3$ 5. $a_1 = -3, a_{n+1} = a_n + n$	
	6. $a_1 = 0, a_{n+1} = -2a_n - 4$ 7. $a_1 = 1, a_2 = 2, a_{n+2} = 4a_{n+1} - 3$	Ba _n
	Find the first three iterates of each function for the given initial value.	
	8. $f(x) = 3x - 4$, $x_0 = 3$ 9. $f(x) = -2x + 5$, $x_0 = 2$ 10. $f(x) = x^2 + 2$, $x_0 = -2x + 5$, $x_0 = 2$ 10. $f(x) = x^2 + 2$, $x_0 = -2x + 5$,	-1
Application	BANKING For Exercises 11 and 12, use the following information. Rita has deposited \$1000 in a bank account. At the end of each year, the bank pointerest to her account in the amount of 5% of the balance, but then takes out a \$	osts
	annual fee.	p 10
	 annual fee. 11. Let b₀ be the amount Rita deposited. Write a recursive equation for the balar b_n in her account at the end of <i>n</i> years. 	nce



Practice and Apply

Homework Help

For Exercises	See Examples
13-30	1-2
31-39	3

Extra Practice See page 853.

Find the first five terms of each sequence.

- **13.** $a_1 = -6$, $a_{n+1} = a_n + 3$ **14.** $a_1 = 13, a_{n+1} = a_n + 5$ **16.** $a_1 = 6, a_{n+1} = a_n + n + 3$ **15.** $a_1 = 2, a_{n+1} = a_n - n$ 17. $a_1 = 9, a_{n+1} = 2 a_n - 4$ **18.** $a_1 = 4, a_{n+1} = 3a_n - 6$ **19.** $a_1 = -1, a_2 = 5, a_{n+1} = a_n + a_{n-1}$ **20.** $a_1 = 4, a_2 = -3, a_{n+2} = a_{n+1} + 2a_n$ **21.** $a_1 = \frac{7}{2}, a_{n+1} = \frac{n}{n+1} \cdot a_n$ **22.** $a_1 = \frac{3}{4}, a_{n+1} = \frac{n^2 + 1}{n} \cdot a_n$ **23.** If $a_0 = 7$ and $a_{n+1} = a_n + 12$ for $n \ge 0$, find the value of a_5 .
- **24.** If $a_0 = 1$ and $a_{n+1} = -2.1$ for $n \ge 0$, then what is the value of a_4 ?

GEOMETRY For Exercises 25 and 26, use the following information.

Join two 1-unit by 1-unit squares to form a rectangle. Next, draw a larger square along a long side of the rectangle. Continue this process of drawing a square along a long side of the rectangle formed at the previous step.



Most real estate agents are independent businesspeople who earn their income from commission.

🖢 Online Research

To learn more about a career in real estate, visit: www.algebra2.com/ careers



- **25.** Write the sequence of the lengths of the sides of the squares you added at each step. Begin the sequence with the lengths of the sides of the two original squares.
- 26. Identify the sequence in Exercise 25.
- **Real Estate Agent** **27. LOANS** The Cruz family is taking out a mortgage loan for \$100,000 to buy a house. Their monthly payment is \$670.70.70 $b_n = 1.006 b_{n-1} - 678.79$ describes the balance left on the loan after n payments. Find the balances of the loan after each of the first eight payments.





- **28.** Write a sequence of the first five triangular numbers.
- **29.** Write a recursive formula for the *n*th triangular number t_n .
- 30. What is the 200th triangular number?

www.algebra2.com/self check quiz

Lesson 11-6 Recursion and Special Sequences 609

Find the first three iterates of each function for the given initial value.

31. $f(x) = 9x - 2, x_0 = 2$	32. $f(x) = 4x - 3, x_0 = 2$
33. $f(x) = 3x + 5, x_0 = -4$	34. $f(x) = 5x + 1$, $x_0 = -1$
35. $f(x) = 2x^2 - 5$, $x_0 = -1$	36. $f(x) = 3x^2 - 4$, $x_0 = 1$
37. $f(x) = 2x^2 + 2x + 1, x_0 = \frac{1}{2}$	38. $f(x) = 3x^2 - 3x + 2$, $x_0 = \frac{1}{3}$

- **39. ECONOMICS** If the rate of inflation is 2%, the cost of an item in future years can be found by iterating the function c(x) = 1.02x. Find the cost of a \$70 portable stereo in four years if the rate of inflation remains constant.
- **40. CRITICAL THINKING** Are there a function f(x) and an initial value x_0 such that the first three iterates, in order, are 4, 4, and 7? If so, state such a function and initial value. If not, explain.
- 41. WRITING IN MATH Answer the question that was posed at the beginning of the lesson.

How is the Fibonacci sequence illustrated in nature?

Include the following in your answer:

- the 13th term in the Fibonacci sequence, with an explanation of what it tells you about the plant described, and
- an explanation of why the Fibonacci sequence is neither arithmetic nor geometric.
- Standardized A B C D



Maintain Your Skills

Mixed Review Find the sum of each infinite geometric series, if it exists. (Lesson 11-5) **45.** $\frac{1}{8} + \frac{1}{32} + \frac{1}{128} + \dots$ **46.** $4 - \frac{8}{3} + \frac{16}{9} + \dots$ **44.** 9 + 6 + 4 + ... Find the sum of each geometric series. (Lesson 11-4) **48.** $3 + 1 + \frac{1}{3} + \dots$ to 7 terms **47.** $2 - 10 + 50 - \dots$ to 6 terms **49. GEOMETRY** The area of rectangle *ABCD* В is $6x^2 + 38x + 56$ square units. Its width is 2x + 8 units. What is the length of the rectangle? 2x + 8 (Lesson 5-3) D C Getting Ready for BASIC SKILL Evaluate each expression. the Next Lesson 50. $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ **51.** $7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ **52.** $\frac{4 \cdot 3}{2 \cdot 1}$ **54.** $\frac{9 \cdot 8 \cdot 7 \cdot 6}{4 \cdot 3 \cdot 2 \cdot 1}$ **55.** $\frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$ **53.** $\frac{6 \cdot 5 \cdot 4}{3 \cdot 2 \cdot 1}$





Algebra Activity

A Follow-Up of Lesson 11-6

Fractals

Fractals are sets of points that often involve intricate geometric shapes. Many fractals have the property that when small parts are magnified, the detail of the fractal is not lost. In other words, the magnified part is made up of smaller copies of itself. Such fractals can be constructed recursively.

You can use isometric dot paper to draw stages of the construction of a fractal called the *von Koch snowflake*.

Stage 1 Draw an equilateral triangle with sides of length 9 units on the dot paper.

Stage 2 Now remove the middle third of each side of the triangle from Stage 1 and draw the other two sides of an equilateral triangle pointing outward.





Imagine continuing this process indefinitely. The von Koch snowflake is the shape that these stages approach.

Model and Analyze

1. Copy and complete the table. Draw Stage 3, if necessary.

Stage	1	2	3	4
Number of Segments	3	12		
Length of each Segment	9	3		
Perimeter	27	36		

- **2.** Write recursive formulas for the number s_n of segments in Stage *n*, the length ℓ_n of each segment in Stage *n*, and the perimeter P_n of Stage *n*.
- **3.** Write nonrecursive formulas for s_n , ℓ_n , and P_n .
- 4. What is the perimeter of the von Koch snowflake? Explain.
- **5.** Explain why the area of the von Koch snowflake can be represented by the infinite series $\frac{81\sqrt{3}}{4} + \frac{27\sqrt{3}}{4} + 3\sqrt{3} + \frac{4\sqrt{3}}{3} + \dots$

CONTENTS

- 6. Find the sum of the series in Exercise 5. Explain your steps.
- **7.** Do you think the results of Exercises 4 and 6 are contradictory? Explain.

11-7 The Binomial Theorem

What You'll Learn

- Use Pascal's triangle to expand powers of binomials.
- Use the Binomial Theorem to expand powers of binomials.

Vocabulary

- Pascal's triangle
- Binomial Theorem
- factorial

How does a power of a binomial describe the numbers of boys and girls in a family?

According to the U.S. Census Bureau, ten percent of families have three or more children. If a family has four children, there are six sequences of births of boys and girls that result in two boys and two girls. These sequences are listed below.

BBGG	BGBG	BGGB	GBBG	GBGB	GGBB

••• **PASCAL'S TRIANGLE** You can use the coefficients in powers of binomials to count the number of possible sequences in situations such as the one above. Remember that a binomial is a polynomial with two terms. Expand a few powers of the binomial b + g.

$(b + g)^0 =$	$1b^{0}g^{0}$
$(b + g)^1 =$	$1b^1g^0 + 1b^0g^1$
$(b + g)^2 =$	$1b^2g^0 + 2b^1g^1 + 1b^0g^1$
$(b + g)^3 =$	$1b^3g^0 + 3b^2g^1 + 3b^1g^2 + 1b^0g^3$
$(b + g)^4 =$	$1b^4g^0 + 4b^3g^1 + 6b^2g^2 + 4b^1g^3 + 1b^0g^4$

The coefficient 6 of the b^2g^2 term in the expansion of $(b + g)^4$ gives the number of sequences of births that result in two boys and two girls. As another example, the coefficient 4 of the b^1g^3 term gives the number of sequences with one boy and 3 girls.

Here are some patterns that can be seen in any binomial expansion of the form $(a + b)^n$.

- **1.** There are n + 1 terms.
- **2.** The exponent *n* of $(a + b)^n$ is the exponent of *a* in the first term and the exponent of *b* in the last term.
- **3.** In successive terms, the exponent of *a* decreases by one, and the exponent of *b* increases by one.
- **4.** The sum of the exponents in each term is *n*.
- **5.** The coefficients are symmetric. They increase at the beginning of the expansion and decrease at the end.

The coefficients form a pattern that is often displayed in a triangular formation. This is known as **Pascal's triangle**. Notice that each row begins and ends with 1. Each coefficient is the sum of the two coefficients above it in the previous row.



More About. .



Pascal's Triangle •···· Although he did not discover it, Pascal's triangle is named for the French mathematician Blaise Pascal (1623–1662).

Example 🚺 Use Pascal's Triangle

Expand $(x + y)^7$.

Write two more rows of Pascal's triangle.

1 6 15 20 15 6 1 1 7 21 35 35 21 7 1

Use the patterns of a binomial expansion and the coefficients to write the expansion of $(x + y)^7$.

$$(x + y)^7 = 1x^7y^0 + 7x^6y^1 + 21x^5y^2 + 35x^4y^3 + 35x^3y^4 + 21x^2y^5 + 7x^1y^6 + 1x^0y^7$$

= $x^7 + 7x^6y + 21x^5y^2 + 35x^4y^3 + 35x^3y^4 + 21x^2y^5 + 7xy^6 + y^7$

THE BINOMIAL THEOREM Another way to show the coefficients in a binomial expansion is to write them in terms of the previous coefficients.

$(a + b)^0$					1	Eliminate common
$(a + b)^1$				1	$\frac{1}{1}$	shown in color.
$(a + b)^2$			1		$\frac{2}{1}$ $\frac{2 \cdot 1}{1 \cdot 2}$	<u>-</u> !
$(a + b)^3$		1		$\frac{3}{1}$	$\frac{3\cdot 2}{1\cdot 2}$	$\frac{3\cdot 2\cdot 1}{1\cdot 2\cdot 3}$
$(a + b)^4$	1		$\frac{4}{1}$		$\frac{4\cdot 3}{1\cdot 2} \qquad \frac{4\cdot 3}{1\cdot 2}$	$\frac{4 \cdot 3 \cdot 2 \cdot 1}{1 \cdot 2 \cdot 3 \cdot 4}$

This pattern provides the coefficients of $(a + b)^n$ for any nonnegative integer *n*. The pattern is summarized in the **Binomial Theorem**.

Key Concept If *n* is a nonnegative integer, then $(a + b)^n = 1a^nb^0 + \frac{n}{1}a^{n-1}b^1 + \frac{n(n-1)}{1\cdot 2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}b^3 + ... + 1a^0b^n.$

Example 2 Use the Binomial Theorem

Expand $(a - b)^6$.

The expansion will have seven terms. Use the sequence $1, \frac{6}{1}, \frac{6 \cdot 5}{1 \cdot 2}, \frac{6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3}$ to find

the coefficients for the first four terms. Then use symmetry to find the remaining coefficients.

$$(a-b)^{6} = 1a^{6} (-b)^{0} + \frac{6}{1}a^{5} (-b)1 + \frac{6 \cdot 5}{1 \cdot 2}a^{4} (-b)^{2} + \frac{6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3}a^{3} (-b)^{3} + \dots + 1a^{0} (-b)^{6}$$
$$= a^{6} - 6a^{5}b + 15a^{4}b^{2} - 20a^{3}b^{3} + 15a^{2}b^{4} - 6ab^{5} + b^{6}$$

Notice that in terms having the same coefficients, the exponents are reversed, as in $15a^4b^2$ and $15a^2b^4$.

The factors in the coefficients of binomial expansions involve special products called **factorials**. For example, the product $4 \cdot 3 \cdot 2 \cdot 1$ is written 4! and is read 4 *factorial*. In general, if *n* is a positive integer, then $n! = n(n - 1)(n - 2)(n - 3) \dots 2 \cdot 1$. *By definition*, 0! = 1.

💭 www.algebra2.com/extra_examples

Study Tip

Graphing Calculators On a TI-83 Plus, the

PRB menu.

factorial symbol, !, is

located on the MATH

Example 3 Factorials Evaluate $\frac{8!}{3!5!}$. $\frac{8!}{3!5!} = \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 - 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1 \cdot 5 \cdot 4 - 3 \cdot 2 \cdot 1}$ Note that $8! = 8 \cdot 7 \cdot 6 \cdot 5!$, so $\frac{8!}{3!5!} = \frac{8 \cdot 7 \cdot 6 \cdot 5!}{3!5!}$ or $\frac{8 \cdot 7 \cdot 6}{3 \cdot 2 \cdot 1}$ $= \frac{8 \cdot 7 \cdot 6}{3 \cdot 2 \cdot 1}$ or 56

An expression such as $\frac{6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3}$ in Example 2 can be written as a quotient of factorials. In this case, $\frac{6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3} = \frac{6!}{3!3!}$. Using this idea, you can rewrite the

expansion of $(a + b)^6$ using factorials.

$$(a+b)^6 = \frac{6!}{6!0!}a^6b^0 + \frac{6!}{5!1!}a^5b^1 + \frac{6!}{4!2!}a^4b^2 + \frac{6!}{3!3!}a^3b^3 + \frac{6!}{2!4!}a^2b^4 + \frac{6!}{1!5!}a^1b^5 + \frac{6!}{0!6!}a^0b^6$$

You can also write this series using sigma notation.

$$(a + b)^{6} = \sum_{k=0}^{6} \frac{6!}{(6 - k)!k!} a^{6 - k} b^{k}$$

In general, the Binomial Theorem can be written both in factorial notation and in sigma notation.

Key Concept
 Binomial Theorem, Factorial Form

$$(a + b)^n = \frac{n!}{n!0!} a^n b^0 + \frac{n!}{(n-1)!1!} a^{n-1} b^1 + \frac{n!}{(n-2)!2!} a^{n-2} b^2 + \dots + \frac{n!}{0!n!} a^0 b^n$$
 $= \sum_{k=0}^n \frac{n!}{(n-k)!k!} a^{n-k} b^k$

Example 4 Use a Factorial Form of the Binomial Theorem Expand $(2x + y)^5$.

$$(2x + y)^5 = \sum_{k=0}^{5} \frac{5!}{(5 - k)!k!} (2x)^{5-k} y^k$$
Binomial Theorem, factorial form
$$= \frac{5!}{5!0!} (2x)^5 y^0 + \frac{5!}{4!1!} (2x)^4 y^1 + \frac{5!}{3!2!} (2x)^3 y^2 + \frac{5!}{2!3!} (2x)^2 y^3 + \frac{5!}{1!4!} (2x)^1 y^4 + \frac{5!}{0!5!} (2x)^0 y^5$$
Let $k = 0, 1, 2, 3, 4, \text{ and } 5.$
$$= \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot 1} (2x)^5 + \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{4 \cdot 3 \cdot 2 \cdot 1 \cdot 1} (2x)^4 y + \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1 \cdot 2 \cdot 1} (2x)^3 y^2 + \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{1 \cdot 4 \cdot 3 \cdot 2 \cdot 1} (2x)^2 y^4 + \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{1 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} y^5$$
$$= 32x^5 + 80x^4y + 80x^3y^2 + 40x^2y^3 + 10xy^4 + y^5$$
Simplify.

Sometimes you need to know only a particular term of a binomial expansion. Note that when the Binomial Theorem is written in sigma notation, k = 0 for the first term, k = 1 for the second term, and so on. In general, the value of k is always one less than the number of the term you are finding.

Study Tip

Missing Steps If you don't understand a step like $\frac{6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3} = \frac{6!}{3!3!}$, work it out on a piece of scrap paper. $\frac{6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3} = \frac{6 \cdot 5 \cdot 4 \cdot 3!}{1 \cdot 2 \cdot 3 \cdot 3!}$ $= \frac{6!}{3!3!}$



Example 5 Find a Particular Term

Find the fifth term in the expansion of $(p + q)^{10}$.

First, use the Binomial Theorem to write the expansion in sigma notation.

$$(p+q)^{10} = \sum_{k=0}^{10} \frac{10!}{(10-k)!k!} p^{10-k} q^k$$

In the fifth term,
$$k = 4$$
.

$$\frac{10!}{(10-k)!k!}p^{10-k}q^{k} = \frac{10!}{(10-4)!4!}p^{10-4}q^{4} \quad k = 4$$

$$= \frac{10 \cdot 9 \cdot 8 \cdot 7}{4 \cdot 3 \cdot 2 \cdot 1}p^{6}q^{4} \qquad \frac{10!}{6!4!} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6!}{6!4!} \text{ or } \frac{10 \cdot 9 \cdot 8 \cdot 7}{4 \cdot 3 \cdot 2 \cdot 1}$$

$$= 210p^{6}q^{4} \qquad \text{Simplify.}$$

Check for Understanding

Concept Check	 List the coefficients in the row of Pascal's triangle corresponding to n = 8. Identify the coefficient of a^{n - 1}b in the expansion of (a + b)ⁿ. 		
	3. OPEN ENDED Write a power of a binomial for which the first term of the expansion is $625x^4$.		
Guided Practice	Evaluate each expression. 4. 8!	5. $\frac{13!}{9!}$	6. $\frac{12!}{2!10!}$
	Expand each power. 7. $(p + q)^5$	8. $(t+2)^6$	9. $(x - 3y)^4$
	Find the indicated term of a 10 . fourth term of $(a + b)^8$	each expansion. 11. fifth term	n of $(2a + 3b)^{10}$

Application 12. SCHOOL Mr. Hopkins is giving a five-question true-false quiz. How many ways could a student answer the questions with three trues and two falses?

Practice and Apply

Homework Help		Evaluate each expression.	
For Exercises	See Examples	13. 9!	
13–18	3	16 $\frac{7!}{10}$	
19–33	1, 2, 4	4!	
34-41	5	Expand each power.	
Extra P	ractice	19. $(a - b)^3$	

See page 853.

13.	9!	14.	13!
16.	<u>7!</u> <u>4!</u>	17.	$\frac{12!}{8!4!}$
Exp	and each power.		
19.	$(a - b)^3$	20.	(m
22.	$(m - a)^5$	23.	(x -
25.	$(2b - x)^4$	26.	(2a
28.	$(3x+2y)^4$	29.	$\left(\frac{a}{2}\right)$

- $(+ n)^4$ $(+ 3)^{5}$ $(+ b)^{6}$ $(+2)^{5}$ 31. **GEOMETRY** Write an expanded expression
- **21.** $(r + s)^8$ **24.** $(a - 2)^4$ **27.** $(3x - 2y)^5$ **30.** $\left(3 + \frac{m}{3}\right)^5$

15. $\frac{9!}{7!}$ **18.** $\frac{14!}{5!9!}$



www.algebra2.com/self_check_quiz



for the volume of the cube at the right.

Lesson 11-7 The Binomial Theorem 615



Pascal's triangle displays many patterns. Visit www.algebra2.com/ webquest to continue work on your WebQuest project. **32. GAMES** The diagram shows the board for a game in which ball bearings are dropped down a chute. A pattern of nails and dividers causes the bearings to take various paths to the sections at the bottom. For each section, how many paths through the board lead to that section?



33. INTRAMURALS Ofelia is taking ten shots in the intramural free-throw shooting competition. How many sequences of makes and misses are there that result in her making eight shots and missing two?

Find the indicated term of each expansion.

- **34.** sixth term of $(x y)^9$
- **36.** fourth term of $(x + 2)^7$
- **38.** fifth term of $(2a + 3b)^{10}$
- **40.** fourth term of $\left(x + \frac{1}{3}\right)^7$
- **37.** fifth term of $(a 3)^8$ **39.** fourth term of $(2x + 3y)^9$

35. seventh term of $(x + y)^{12}$

- **41.** sixth term of $\left(x \frac{1}{2}\right)^{10}$
- **42. CRITICAL THINKING** Explain why $\frac{12!}{7!5!} + \frac{12!}{6!6!} = \frac{13!}{7!6!}$ without finding the value of any of the expressions.
- **43.** WRITING IN MATH Answer the question that was posed at the beginning of the lesson.

How does a power of a binomial describe the numbers of boys and girls in a family?

Include the following in your answer:

- the expansion of $(b + g)^5$ and what it tells you about sequences of births of boys and girls in families with five children, and
- an explanation of how to find a formula for the number of sequences of births that have exactly *k* girls in a family of *n* children.



- 44. Which of the following represents the values of *x* that are solutions of the inequality $x^2 < x + 20$?
 - (A) x > -4(B) x < 5
 - \bigcirc -5 < x < 4
 - **D** -4 < x < 5
 - **45.** If four lines intersect as shown in the figure at the right, x + y =
 - **A** 70.
 - **B** 115.
 - **C** 140.
 - D It cannot be determined from the information given.





Maintain Your Skills

Mixed Review Find the first five terms of each sequence. (Lesson 11-6)

46.
$$a_1 = 7, a_{n+1} = a_n - 2$$
 47. $a_1 = 3, a_{n+1} = 2a_n$

48. CLOCKS The spring in Juanita's old grandfather clock is broken. When you try to set the pendulum in motion by holding it against the wall of the clock and letting go, it follows a swing pattern of 25 centimeters, 20 centimeters, 16 centimeters, and so on until it comes to rest. What is the total distance the pendulum swings before coming to rest? (Lesson 11-5)

Express each logarithm in terms of common logarithms. Then approximate its value to four decimal places. (Lesson 10-4)

49.
$$\log_2 5$$
 50. $\log_3 10$ **51.** $\log_5 8$

Determine any vertical asymptotes and holes in the graph of each rational function. (Lesson 9-3)

52.
$$f(x) = \frac{1}{x^2 + 5x + 6}$$
 53. $f(x) = \frac{x + 2}{x^2 + 3x - 4}$ **54.** $f(x) = \frac{x^2 + 4x + 3}{x + 3}$

Without writing the equation in standard form, state whether the graph of each equation is a parabola, circle, ellipse, or hyperbola. (Lesson 8-6)

55.
$$x^2 - 6x - y^2 - 3 = 0$$
 56. $4y - x + y^2 = 1$

Determine whether each pair of functions are inverse functions. (Lesson 7-8)

57.
$$f(x) = x + 3$$

 $g(x) = x - 3$
58. $f(x) = 2x + 1$
 $g(x) = \frac{x + 1}{2}$

Getting Ready for **PREREQUISITE SKILL** State whether each statement is *true* or *false* when n = 1. the Next Lesson Explain. (To review evaluating expressions, see Lesson 1-1.)

59.
$$1 = \frac{n(n+1)}{2}$$

60. $1 = \frac{(n+1)(2n+1)}{2}$
61. $1 = \frac{n^2(n+1)^2}{4}$
62. $3^n - 1$ is even.

Practice Quiz 2

Lessons 11-4 through 11-7

- 1

Find the sum of each geometric series. (Lessons 11-4 and 11-5)

1.
$$a_1 = 5, r = 3, n = 12$$

2. $\sum_{n=1}^{\infty} 2(-3)^{n-1}$
3. $\sum_{n=1}^{\infty} 8 \left(\frac{2}{3}\right)^{n-1}$
4. $5 + 1 + \frac{1}{5} + \dots$

Find the first five terms of each sequence. (Lesson 11-6)

5.
$$a_1 = 1, a_{n+1} = 2a_n + 3$$

6. $a_1 = 2, a_{n+1} = a_n + 2n$

7. Find the first three iterates of the function f(x) = -3x + 2 for an initial value of $x_0 = -1$. (Lesson 11-6)

Expand each power. (Lesson 11-7) 8. $(3x + y)^5$

9. $(a + 2)^6$

10. Find the fifth term of the expansion of $(2a + b)^9$. (Lesson 11-7)



Proof and Mathematical Induction

What You'll Learn

- Prove statements by using mathematical induction.
- Disprove statements by finding a counterexample.

Vocabulary

11-8

- mathematical induction
- inductive hypothesis

How does the concept of a ladder help you prove statements about numbers?

Imagine the positive integers as a ladder that goes upward forever. You know that you cannot leap to the top of the ladder, but you can stand on the first step, and no matter which step you are on, you can always climb one step higher. Is there any step you cannot reach?



MATHEMATICAL INDUCTION Mathematical induction is used to prove statements about positive integers. An induction proof consists of three steps.

Key Concept

Mathematical Induction

- **Step 1** Show that the statement is true for some integer *n*.
- **Step 2** Assume that the statement is true for some positive integer k, where $k \ge n$. This assumption is called the **inductive hypothesis**.
- **Step 3** Show that the statement is true for the next integer k + 1.

Example 🚺 Summation Formula

Prove that the sum of the squares of the first *n* positive integers is $\frac{n(n+1)(2n+1)}{6}$. That is, prove that $1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$. **Step 1** When n = 1, the left side of the given equation is 1^2 or 1. The right side is $\frac{1(1+1)[2(1)+1]}{6}$ or 1. Thus, the equation is true for n = 1. **Step 2** Assume $1^2 + 2^2 + 3^2 + ... + k^2 = \frac{k(k+1)(2k+1)}{6}$ for a positive integer *k*. Show that the given equation is true for n = k + 1. Step 3 $1^2 + 2^2 + 3^2 + \dots + k^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2$ Add $(k+1)^2$ to each side to each side. $= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6}$ Add. $= \frac{(k+1)[k(2k+1)+6(k+1)]}{6}$ Factor. $= \frac{(k+1)[2k^2+7k+6]}{6}$ Simplify. $=\frac{(k+1)(k+2)(2k+3)}{6}$ Factor. $= \frac{(k+1)[(k+1)+1][2(k+1)+1]}{6}$



Step 1 In many cases, it will be helpful to let n = 1.



The last expression on page 618 is the right side of the equation to be proved, where *n* has been replaced by k + 1. Thus, the equation is true for n = k + 1.

This proves that $1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$ for all positive integers *n*.

Example 2 Divisibility

Prove that $7^n - 1$ is divisible by 6 for all positive integers *n*.

- **Step 1** When n = 1, $7^n 1 = 7^1 1$ or 6. Since 6 is divisible by 6, the statement is true for n = 1.
- **Step 2** Assume that $7^k 1$ is divisible by 6 for some positive integer *k*. This means that there is a whole number *r* such that $7^k 1 = 6r$.
- **Step 3** Show that the statement is true for n = k + 1.

 $\begin{array}{ll} 7^k-1=6r & \mbox{Inductive hypothesis} \\ 7^k=6r+1 & \mbox{Add 1 to each side.} \\ 7(7^k)=7(6r+1) & \mbox{Multiply each side by 7.} \\ 7^{k+1}=42r+7 & \mbox{Simplify.} \\ 7^{k+1}-1=42r+6 & \mbox{Subtract 1 from each side.} \\ 7^{k+1}-1=6(7r+1) & \mbox{Factor.} \end{array}$

Since *r* is a whole number, 7r + 1 is a whole number. Therefore, $7^{k+1} - 1$ is divisible by 6. Thus, the statement is true for n = k + 1.

This proves that $7^n - 1$ is divisible by 6 for all positive integers *n*.

.....

COUNTEREXAMPLES Of course, not every formula that you can write is true. A formula that works for a few positive integers may not work for *every* positive integer. You can show that a formula is not true by finding a *counterexample*. This often involves trial and error.

Example 3 Counterexample

Find a counterexample for the formula $1^4 + 2^4 + 3^4 + ... + n^4 = 1 + (4n - 4)^2$. Check the first few positive integers.

	n	Left Side of Formula	Right Side of Formula	
	1	1 ⁴ or 1	$1 + [4(1) - 4]^2 = 1 + 0^2 \text{ or } 1$	true
	2	$1^4 + 2^4 = 1 + 16 \text{ or } 17$	$1 + [4(2) - 4]^2 = 1 + 4^2$ or 17	true
J	3	$1^4 + 2^4 + 3^4 = 1 + 16 + 81$ or 98	$1 + [4(3) - 4]^2 = 1 + 64$ or 65	false

The value n = 3 is a counterexample for the formula.

Check for Understanding

Concept Check **1. Describe** some of the types of statements that can be proved by using mathematical induction.

- 2. Explain the difference between mathematical induction and a counterexample.
- **3. OPEN ENDED** Write an expression of the form $b^n 1$ that is divisible by 2 for all positive integers *n*.

www.algebra2.com/extra_examples



Study Tip

Reading Math

One of the meanings of *counter* is *to oppose*, so a counterexample is an example that opposes a hypothesis.

Guided Practice Prove that each statement is true for all positive integers.

4. $1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$ **5.** $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots + \frac{1}{2^n} = 1 - \frac{1}{2^n}$ **6.** $4^n - 1$ is divisible by 3.**7.** $5^n + 3$ is divisible by 4.

Find a counterexample for each statement.

```
8. 1 + 2 + 3 + \ldots + n = n^2
```

9. $2^n + 2n$ is divisible by 4.

Application 10. PARTIES Suppose that each time a new guest arrives at a party, he or she shakes hands with each person already at the party. Prove that after *n* guests have arrived, a total of $\frac{n(n-1)}{2}$ handshakes have taken place.

Practice and Apply

Homework Help			
For Exercises	See Examples		
11-23, 31	1		
24	1, 2		
25-30	3		

Extra Practice See page 853.

More About. .



Architecture •·····

The Vietnam Veterans Memorial lists the names of 58,220 deceased or missing soldiers. Source: National Parks Service

- Prove that each statement is true for all positive integers.
- **11.** $1 + 5 + 9 + \dots + (4n 3) = n(2n 1)$ **12.** $2 + 5 + 8 + \dots + (3n - 1) = \frac{n(3n + 1)}{2}$ **13.** $1^3 + 2^3 + 3^3 + \dots + n^3 = \frac{n^2(n + 1)^2}{4}$
- **14.** $1^2 + 3^2 + 5^2 + \ldots + (2n 1)^2 = \frac{n(2n 1)(2n + 1)}{2n}$
- **15.** $\frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \dots + \frac{1}{3^n} = \frac{1}{2} \left(1 \frac{1}{3^n} \right)$ **16.** $\frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \dots + \frac{1}{4^n} = \frac{1}{3} \left(1 - \frac{1}{4^n} \right)$
- **17.** $8^n 1$ is divisible by 7.
- **18.** $9^n 1$ is divisible by 8.
- **19.** $12^n + 10$ is divisible by 11.
- **20.** $13^n + 11$ is divisible by 12.
- 21. ARCHITECTURE A memorial being constructed in a city park will be a brick wall, with a top row of six gold-plated bricks engraved with the names of six local war veterans. Each row has two more bricks than the row above it. Prove that the number of bricks in the top *n* rows is $n^2 + 5n$.



22. GEOMETRIC SERIES Use mathematical induction to prove the formula $a_1 + a_1r + a_1r^2 + ... + a_1r^{n-1} = \frac{a_1(1 - r^n)}{1 - r}$ for the sum of a finite geometric series.

23. ARITHMETIC SERIES Use mathematical induction to prove the formula

 $a_1 + (a_1 + d) + (a_1 + 2d) + \dots + [a_1 + (n-1)d] = \frac{n}{2}[2a_1 + (n-1)d]$ for the sum of an arithmetic series.

24. PUZZLES Show that a 2^{*n*} by 2^{*n*} checkerboard with the top right square missing can always be covered by nonoverlapping L-shaped tiles like the one at the right.





Find a counterexample for each statement.

25.
$$1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(3n-1)}{2}$$

26. $1^3 + 3^3 + 5^3 + \dots + (2n-1)^3 = 12n^3 - 23n^2 + 12n^3$

.....

- **27.** $3^n + 1$ is divisible by 4.
- **28.** $2^n + 2n^2$ is divisible by 4.
- **29.** $n^2 n + 11$ is prime.
- **30.** $n^2 + n + 41$ is prime.
- 31. CRITICAL THINKING Refer to Example 2. Explain how to use the Binomial Theorem to show that $7^n - 1$ is divisible by 6 for all positive integers *n*.
- 32. WRITING IN MATH Answer the question that was posed at the beginning of the lesson.

How does the concept of a ladder help you prove statements about numbers?

Include the following in your answer:

- an explanation of which part of an inductive proof corresponds to stepping onto the bottom step of the ladder, and
- an explanation of which part of an inductive proof corresponds to climbing from one step on the ladder to the next.



33.
$$\frac{x - \frac{4}{x}}{1 - \frac{4}{x} + \frac{4}{x^2}} =$$

34. Quantitative Comparison

Compare the quantity in Column A and the quantity in Column B. Then determine whether:

- A the quantity in Column A is greater,
- **B** the quantity in Column B is greater,
- **C** the two quantities are equal, or
- **D** the relationship cannot be determined from the information given. PQRS is a square.



Maintain Your Skills

Mixed Review Expand each power. (Lesson 11-7) **36.** $(a - b)^7$ 35. $(x + y)^6$

37.
$$(2x + y)^8$$

Find the first three iterates of each function for the given initial value. (Lesson 11-6)

38. $f(x) = 3x - 2, x_0 = 2$

- **39.** $f(x) = 4x^2 2$, $x_0 = 1$
- **40. BIOLOGY** Suppose an amoeba divides into two amoebas once every hour. How long would it take for a single amoeba to become a colony of 4096 amoebas? (Lesson 10-2)

Solve each equation. Check your solutions. (Lesson 9-6)

41.
$$\frac{1}{y+1} - \frac{3}{y-3} =$$

42.
$$\frac{6}{a-7} = \frac{a-49}{a^2-7a} + \frac{1}{a}$$

www.algebra2.com/self_check_quiz



2

Study Guide and Review

Vocabulary and Concept Check

arithmetic means (p. 580) geometric means (p. 590) arithmetic sequence (p. 578) geometric sequence (p. 588) arithmetic series (p. 583) geometric series (p. 594) Binomial Theorem (p. 613) index of summation (p. 585) common difference (p. 578) inductive hypothesis (p. 618) common ratio (p. 588) infinite geometric series (p. 599) factorial (p. 613) iteration (p. 608) Fibonacci sequence (p. 606) mathematical induction (p. 618) partial sum (p. 599) Pascal's triangle (p. 612) recursive formula (p. 606) sequence (p. 578) series (p. 583) sigma notation (p. 585) term (p. 578)

Choose the term from the list above that best completes each statement.

- **1.** A(n) ______ of an infinite series is the sum of a certain number of terms.
- 2. If a sequence has a common ratio, then it is a(n) _
- **3.** Using _____, the series 2 + 5 + 8 + 11 + 14 can be written as $\sum (3n 1)$.
- 4. Eleven and 17 are the two _____ between 5 and 23 in the sequence 5, 11, 17, 23.
- 5. Using the _____, $(a 2)^4$ can be expanded to $a^4 8a^3 + 24a^2 32a + 16$.
- 6. The ______ of the sequence 3, 2, $\frac{4}{3}$, $\frac{8}{9}$, $\frac{16}{27}$ is $\frac{2}{3}$
- 7. The _____ 11 + 16.5 + 22 + 27.5 + 33 has a sum of 110.
- 8. A(n) _____ is expressed as $n! = n(n-1)(n-2) \dots 2 \cdot 1$.

Lesson-by-Lesson Review



578-582.

chapter

11-1 Arithmetic Sequences

Concept Summary

7 = d

- An arithmetic sequence is formed by adding a constant to each term to get the next term.
- The *n*th term a_n of an arithmetic sequence with first term a_1 and common difference *d* is given by $a_n = a_1 + (n - 1)d$, where *n* is any positive integer.

Examples

1 Find the 12th term of an arithmetic sequence if $a_1 = -17$ and d = 4.

CONTENTS

$$a_n = a_1 + (n-1)d$$
 Formula for the *n*th terr

$$a_{12} = -17 + (12 - 1)4$$
 $n = 12, a_1 = -17, d = 4$

 $a_{12} = 27$ Simplify.

2 Find the two arithmetic means between 4 and 25.

 $a_n = a_1 + (n-1)d$ Formula for the *n*th term

$$a_4 = 4 + (4 - 1)d$$
 $n = 4, a_1 = 4$

$$25 = 4 + 3d$$
 $a_4 = 25$

The arithmetic means are 4 + 7 or 11 and 11 + 7 or 18.

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Chapter 11 Study Guide and Review

Exercises Find the indicated term of each arithmetic sequence. See Example 2 on p. 579. **9.** $a_1 = 6, d = 8, n = 5$ **10.** $a_1 = -5, d = 7, n = 22$ **11.** $a_1 = 5, d = -2, n = 9$ **12.** $a_1 = -2, d = -3, n = 15$ Find the arithmetic means in each sequence. See Example 4 on page 580. **13.** -7, ? , ? , ? , 9 **14.** 12, ? , ? , 4 **15.** 9, <u>?</u>, <u>?</u>, <u>?</u>, <u>?</u>, -6 **16.** 56, <u>?</u>, <u>?</u>, <u>?</u>, 28

Arithmetic Series

See pages 583-587.

Concept Summary

• The sum S_n of the first *n* terms of an arithmetic series is given by $S_n = \frac{n}{2}[2a_1 + (n-1)d]$ or $S_n = \frac{n}{2}(a_1 + a_n)$.

Example Find S_n for the arithmetic series with $a_1 = 34$, $a_n = 2$, and n = 9.

 $S_n = \frac{n}{2}(a_1 + a_n)$ Sum formula $S_9 = \frac{9}{2}(34 + 2)$ $n = 9, a_1 = 34, a_n = 2$ $S_{0} = 162$ Simplify.

Exercises Find S_n for each arithmetic series. See Examples on pages 584 and 585. **18.** 4 + 10 + 16 + ... + 106 **17.** $a_1 = 12, a_n = 117, n = 36$ **20.** $\sum_{n=2}^{13} (3n+1)$ **19.** $10 + 4 + (-2) + \ldots + (-50)$

Geometric Sequences

See pages 588-592

Concept Summary

- A geometric sequence is one in which each term after the first is found by multiplying the previous term by a common ratio.
- The *n*th term a_n of a geometric sequence with first term a_1 and common ratio *r* is given by $a_n = a_1 \cdot r^{n-1}$, where *n* is any positive integer.

Examples 1 Find the fifth term of a geometric sequence for which $a_1 = 7$ and r = 3.

 $a_n = a_1 \cdot r^{n-1}$ Formula for *n*th term $a_5 = 7 \cdot 3^{5-1}$ $n = 5, a_1 = 7, r = 3$

 $a_5 = 567$ The fifth term is 567.

2 Find two geometric means between 1 and 8.

 $a_n = a_1 \cdot r^{n-1}$ Formula for *n*th term $a_4 = 1 \cdot r^{4-1}$ $n = 4 \text{ and } a_1 = 1$ $8 = r^3$ $a_A = 8$ 2 = r

The geometric means are 1(2) or 2 and 2(2) or 4.



 Exercises
 Find the indicated term of each geometric sequence.

 See Example 2 on page 589.
 21. $a_1 = 2, r = 2, n = 5$ 22. $a_1 = 7, r = 2, n = 4$

 23. $a_1 = 243, r = -\frac{1}{3}, n = 5$ 24. $a_6 \text{ for } \frac{2}{3}, \frac{4}{3}, \frac{8}{3}, \dots$

 Find the geometric means in each sequence.

 See Example 5 on page 590.

 25. $3, \frac{?}{2}, \frac{?}{2}, \frac{?}{2}, \frac{?}{2}, \frac{?}{4}$ 26. $7.5, \frac{?}{2}, \frac{?}{2}, \frac{?}{2}, \frac{?}{2}, \frac{?}{2}, 120$

 27. $8, \frac{?}{2}, \frac{?}{2}, \frac{?}{2}, \frac{?}{2}, \frac{?}{4}$ 28. $5, \frac{?}{2}, \frac{?}{2}, \frac{?}{2}, \frac{?}{2}, 80$



594-598.

Geometric Series

Concept Summary

• The sum S_n of the first *n* terms of a geometric series is given by

$$S_n = \frac{a_1 (1 - r^n)}{1 - r}$$
 or $S_n = \frac{a_1 - a_1 r^n}{1 - r}$, where $r \neq 1$.

Example Find the sum of a geometric series for which $a_1 = 7$, r = 3, and n = 14.

$$S_n = \frac{a_1 - a_1 r^n}{1 - r}$$
 Sum formula
$$S_{14} = \frac{7 - 7 \cdot 3^{14}}{1 - 3} \quad n = 14, a_1 = 7, r = 3$$

$$S_{14} = 16,740,388$$
 Use a calculator.

Exercises Find S_n for each geometric series. See Examples 1 and 3 on pages 595 and 596. **29.** $a_1 = 12, r = 3, n = 5$ **30.** 4 - 2 + 1 - ... to 6 terms **31.** 256 + 192 + 144 + ... to 7 terms **32.** $\sum_{n=1}^{5} \left(-\frac{1}{2}\right)^{n-1}$

1-5 Infinite Geometric Series

See pages Concept Summary 599–604.

• The sum *S* of an infinite geometric series with -1 < r < 1 is given by $S = \frac{a_1}{1 - r}$.

Example Find the sum of the infinite geometric series for which $a_1 = 18$ and $r = -\frac{2}{7}$.

$$S = \frac{a_1}{1 - r}$$
 Sum formula
$$= \frac{18}{1 - \left(-\frac{2}{7}\right)} \quad a_1 = 18, r = -\frac{2}{7}$$
$$= \frac{18}{\frac{9}{7}} \text{ or } 14 \quad \text{Simplify.}$$



Chapter 11 Study Guide and Review

Exercises Find the sum of each infinite geometric series, if it exists. See Example 1 on page 600.

33. $a_1 = 6, r = \frac{11}{12}$ **34.** $\frac{1}{8} - \frac{3}{16} + \frac{9}{32} - \frac{27}{64} + \dots$ **35.** $\sum_{n=1}^{\infty} -2\left(-\frac{5}{8}\right)^{n-1}$

6 Recursion and Special Sequences

See pages **Concept Summary**

- In a recursive formula, each term is formulated from one or more previous terms.
- Iteration is the process of composing a function with itself repeatedly.

Examples

1

606-610

Find the first five terms of the sequence in which $a_1 = 2$ and $a_{n+1} = 2a_n - 1$. Recursive formula $a_{n+1} = 2a_n - 1$ $\begin{array}{c} a_{n+1} = 2a_1 - 1 & n = 1 \\ a_2 = 2(2) - 1 \text{ or } 3 & a_1 = 2 \\ a_{2+1} = 2a_2 - 1 & n = 2 \\ a_3 = 2(3) - 1 \text{ or } 5 & a_2 = 3 \end{array} \qquad \begin{array}{c} a_{3+1} = 2a_3 - 1 & n = 3 \\ a_{4+1} = 2a_3 - 1 & or 9 & a_3 = 5 \\ a_{4+1} = 2a_4 - 1 & or 9 & a_3 = 5 \\ a_{4+1} = 2a_4 - 1 & or 9 & a_4 = 9 \end{array}$

The first five terms of the sequence are 2, 3, 5, 9, and 17.

2 Find the first three iterates of f(x) = -5x - 1 for an initial value of $x_0 = -1$.

The first three iterates are 4, -21, and 104.

Exercises Find the first five terms of each sequence. See Example 1 on page 606.

36. $a_1 = -2, a_{n+1} = a_n + 5$ **37.** $a_1 = 3, a_{n+1} = 4 a_n - 10$ **39.** $a_1 = 1, a_2 = 3, a_{n+2} = a_{n+1} + a_n$ **38.** $a_1 = 2, a_{n+1} = a_n + 3n$

Find the first three iterates of each function for the given initial value. See Example 3 on page 608.

40.	$f(x) = -2x + 3, x_0 = 1$	41. $f(x) = 7x - 4, x_0 = 2$
42.	$f(x) = x^2 - 6, x_0 = -1$	43. $f(x) = -2x^2 - x + 5, x_0 = -2$

The Binomial Theorem

Concept Summary

See pages

612-617

• Pascal's triangle can be used to find the coefficients in a binomial expansion.

• The Binomial Theorem:
$$(a + b)^n = \sum_{k=0}^n \frac{n!}{(n-k)!k!} a^{n-k} b^k$$





Extra Practice, see pages 851–855.
Mixed Problem Solving, see page 872.



Expand $(a - 2b)^4$.

$$(a - 2b)^{4} = \sum_{k=0}^{4} \frac{4!}{(4-k)!k!} a^{4-k} (-2b)^{k}$$
 Binomial Theorem

$$= \frac{4!}{4!0!} a^{4} (-2b)^{0} + \frac{4!}{3!1!} a^{3} (-2b)^{1} + \frac{4!}{2!2!} a^{2} (-2b)^{2} + \frac{4!}{1!3!} a^{1} (-2b)^{3} + \frac{4!}{0!4!} a^{0} (-2b)^{4}$$

$$= a^{4} - 8a^{3}b + 24a^{2}b^{2} - 32ab^{3} + 16b^{4}$$
 Simplify.
Exercises Expand each power. See Examples 1, 2, and 4 on pages 613 and 614.
44. $(x + y)^{3}$ 45. $(x - 2)^{4}$ 46. $(3r + s)^{5}$
Find the indicated term of each expansion. See Example 5 on page 615.
47. fourth term of $(x + 2y)^{6}$ 48. second term of $(4x - 5)^{10}$

Proof and Mathematical Induction See pages **Concept Summary** 618-621. Mathematical induction is a method of proof used to prove statements about the positive integers. Prove $1 + 5 + 25 + ... + 5^{n-1} = \frac{1}{4}(5^n - 1)$ for all positive integers *n*. Example **Step 1** When n = 1, the left side of the given equation is 1. The right side is $\frac{1}{4}(5^1 - 1)$ or 1. Thus, the equation is true for n = 1. **Step 2** Assume that $1 + 5 + 25 + ... + 5^{k-1} = \frac{1}{4}(5^k - 1)$ for some positive integer *k*. **Step 3** Show that the given equation is true for n = k + 1. $1 + 5 + 25 + \ldots + 5^{k-1} + 5^{(k+1)-1} = \frac{1}{4}(5^k - 1) + 5^{(k+1)-1}$ Add $5^{(k+1)-1}$ to each side. $=\frac{1}{4}(5^k-1)+5^k$ Simplify the exponent. $=\frac{5^k-1+4\cdot 5^k}{4}$ Common denominator $=\frac{5\cdot 5^k-1}{4}$ **Distributive Property** $=\frac{1}{4}(5^{k+1}-1)$ $5 \cdot 5^{k} = 5^{k+1}$ The last expression above is the right side of the equation to be proved, where nhas been replaced by k + 1. Thus, the equation is true for n = k + 1. This proves that $1 + 5 + 25 + \ldots + 5^{n-1} = \frac{1}{4}(5^n - 1)$ for all positive integers *n*. **Exercises** Prove that each statement is true for all positive integers. See Examples 1 and 2 on pages 618 and 619.



50. $6^n - 1$ is divisible by 5.

49. $1 + 2 + 4 + \ldots + 2^{n-1} = 2^n - 1$



Vocabulary and Concepts

Choose the correct term to complete each sentence.

- **1.** A sequence in which each term after the first is found by adding a constant to the previous term is called a(n) (*arithmetic*, *geometric*) sequence.
- 2. A (Fibonacci sequence, series) is a sum of terms of a sequence.
- **3.** (*Pascal's triangle , Recursive formulas*) and the Binomial Theorem can be used to expand powers of binomials.

Skills and Applications

- 4. Find the next four terms of the arithmetic sequence 42, 37, 32,
- **5.** Find the 27th term of an arithmetic sequence for which $a_1 = 2$ and d = 6.
- 6. Find the three arithmetic means between -4 and 16.
- **7.** Find the sum of the arithmetic series for which $a_1 = 7$, n = 31, and $a_n = 127$.
- 8. Find the next two terms of the geometric sequence $\frac{1}{81}, \frac{1}{27}, \frac{1}{9}, \dots$
- **9.** Find the sixth term of the geometric sequence for which $a_1 = 5$ and r = -2.
- 10. Find the two geometric means between 7 and 189.
- **11.** Find the sum of the geometric series for which $a_1 = 125$, $r = \frac{2}{5}$, and n = 4.

Find the sum of each series, if it exists.

12.
$$\sum_{k=3}^{15} (14-2k)$$
 13. $\sum_{n=1}^{\infty} \frac{1}{3} (-2)^{n-1}$ **14.** 91 + 85 + 79 + ... + (-29) **15.** 12 + (-6) + 3 - $\frac{3}{2}$ + ...

Find the first five terms of each sequence.

16. $a_1 = 1, a_{n+1} = a_n + 3$

17.
$$a_1 = -3, a_{n+1} = a_n + n^2$$

- **18.** Find the first three iterates of $f(x) = x^2 3x$ for an initial value of $x_0 = 1$.
- **19.** Expand $(2s 3t)^5$.
- **20.** Find the third term of the expansion of $(x + y)^{10}$.

Prove that each statement is true for all positive integers.

- **21.** $1 + 3 + 5 + ... + (2n 1) = n^2$ **22.** $14^n 1$ is divisible by 13.
- **23. DESIGN** A landscaper is designing a wall of white brick and red brick. The pattern starts with 20 red bricks on the bottom row. Each row above it contains 3 fewer red bricks than the preceding row. If the top row contains no red bricks, how many rows are there and how many red bricks were used?
- **24. RECREATION** One minute after it is released, a gas-filled balloon has risen 100 feet. In each succeeding minute, the balloon rises only 50% as far as it rose in the previous minute. How far will the balloon rise in 5 minutes?
- **25. STANDARDIZED TEST PRACTICE** Find the next term in the geometric sequence 8, 6, $\frac{9}{2}$, $\frac{27}{8}$,

(A) $\frac{11}{8}$ (B) $\frac{27}{16}$ 2 www.algebra2.com/chapter_test

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Chapter 11 Practice Test 627

D

Chapter 5 Standardized Test Practice

Part 1 Multiple Choice

Record your answers on the answer sheet provided by your teacher or on a sheet of paper.

- **1.** For all positive integers, let $\boxed{n} = n + g$, where *g* is the greatest factor of *n*, and g < n. If $\boxed{18} = x$, then $\boxed{x} =$
 - (A) 9.(B) 8.(C) 27.(D) 36.
- **2.** If *p* is positive, what percent of 6*p* is 12?



- **3.** A box is 12 units tall, 6 units long, and 8 units wide. A designer is creating a new box that must have the same volume as the first box. If the length and width of the new box are each 50% greater than the length and width of the first box, about how many units tall will the new box be?
 - **(A)** 5.3 **(B)** 6.8
 - C 7.1 D 8.5
- **4.** Which of the following statements must be true when 0 < m < 1?

I
$$\frac{\sqrt{m}}{m} > 1$$
 II $4m < 1$ III $m^2 - m^3 < 0$

- A I only
- B III only
- C I and II only
- D I, II, and III

5. If
$$3kx - \frac{4s}{t} = 3ky$$
, then $x - y = ?$
(A) $-\frac{4s}{3kt}$ (B) $\frac{-4s}{t} + \frac{1}{3k}$
(C) $\frac{4s}{3t} - k$ (D) $\frac{4s}{3kt}$

- **6.** For all $n \neq 0$, what is the slope of the line passing through (3n, -k) and (-n, -k)?
- 7. Which is the graph of the equation $x^2 + (y 4)^2 = 20$?

(A) line (B) parabola

© circle

D ellipse

8.
$$\frac{x - \frac{9}{x}}{1 - \frac{6}{x} + \frac{9}{x^2}} =$$
(A) $\frac{x}{x - 3}$
(B) $\frac{x^2 + 3}{x - 3}$
(C) $\frac{x^2 + 3x}{x - 3}$
(D) $\frac{x^2 + 3x}{(x - 3)^2}$

9. What is the sum of the positive even factors of 30?

A	18	B 30
C	48	D 72

- **10.** If ℓ_1 is parallel to ℓ_2 in the figure, what is the value of *x*?
 - (A) 30

(

- **B** 40
- **C** 70
- **D** 80



The

Princeton — Review Test-Taking Tip

Question 5 Some questions ask you to find the value of an expression. It is often not necessary to find the value of each variable in the expression. For example, to answer Question 5, it is not necessary to find the values of x and y. Isolate the expression x - y on one side of the equation.

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Part 2 Short Response/Grid In

Record your answers on the answer sheet provided by your teacher or on a sheet of paper.

11.
$$AA = \frac{BB}{CC}$$

If *A*, *B*, and *C* are each digits and A = 3B, then what is one possible value of *C*?

12. In the figure, each arc is a semicircle. If *B* is the midpoint of \overline{AD} and *C* is the midpoint of \overline{BD} , what is the ratio of the area of the semicircle \overline{CD} to the area of the semicircle \overline{AD} ?



- **13.** Two people are 17.5 miles apart. They begin to walk toward each other along a straight line at the same time. One walks at the rate of 4 miles per hour, and the other walks at the rate of 3 miles per hour. In how many hours will they meet?
- **14.** If $\frac{x+y}{x} = \frac{5}{4}$, then $\frac{y}{x} =$
- **15.** A car's gasoline tank is $\frac{1}{2}$ full. After adding 7 gallons of gas, the gauge shows that the tank is $\frac{3}{4}$ full. How many gallons does the tank hold?
- **16.** If a = 15 b, what is the value of 3a + 3b?
- **17.** If $x^9 = \frac{45}{y}$ and $x^7 = \frac{1}{5y}$, and x > 0, what is the value of *x*?

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Part 3 Quantitative Comparison

Compare the quantity in Column A and the quantity in Column B. Then determine whether:

- (A) the quantity in Column A is greater,
- **B** the quantity in Column B is greater,
- C the two quantities are equal, or
- **D** the relationship cannot be determined from the information given.

	Column A	Column B
18.	the arithmetic mean	the arithmetic mean
	of three consecutive	of five consecutive
	integers where x is	integers where x is
	the median	the median

19. The area of Square B is equal to nine times the area of Square A.

three times the	the perimeter of
perimeter of Square A	Square B

20.
$$n = n(n + 1)$$
 if *n* is even

$$\binom{n}{n} = n(n-1)$$
 if *n* is odd







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